

HIGHER-ORDER LAGRANGIAN FORMALISM ON GRASSMANN MANIFOLDS

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Abstract

The Lagrangian formalism on a arbitrary non-fibrating manifold is considered. The kinematical description of this generic situation is based on the concept of (higher-order) Grassmann manifolds which is the factorization of the regular velocity manifold to the action of the differential group. Here we introduce in this context the basic concepts of the Lagrangian formalism as Lagrange, Euler-Lagrange and Helmholtz-Sonin forms. These objects come in pairs, namely we have homogeneous objects (defined on the regular velocity manifold) and non-homogeneous objects (defined on the Grassmann manifold). We will establish the connection between the homogeneous objects and their non-homogeneous counterparts. As a result we will conclude that the generic expressions for a variationally trivial Lagrangian and for a locally variational differential equation remain the same as in the fibrating case.

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1 Introduction

The Lagrangian formalism is usually based, from the kinematical point of view, on a fibre bundle with the base manifold and the fibres interpreted as the “space-time” variables and the field variables respectively. As proved in the literature, the natural object associated to such a fibre bundle (from the Lagrangian formalism point of view) is the so-called variational exact sequence [21] - [23]. This sequence contains as distinguished terms the Lagrange, Euler-Lagrange and Helmholtz-Sonin forms, which are the main ingredients of the Lagrangian formalism. Using the exactness property one can obtain, beside a intrinsic geometrical formulation of the Lagrangian formalism, the most general expression of a variationally trivial Lagrangian and the generic form of a locally variational differential equation [12] - [13].

One of the questions not settled in the literature till recently was if the formalism above can be extended to the case when the kinematics of the dynamical system is described by a manifold which is not a fibre bundle. A physical example of this kind is the Minkowski space describing the relativistic particle. For first-order Lagrangian systems a generalization of the Lagrangian formalism covering this case was proposed (see [15] and [11] for a review) based on the notion of Grassmann manifold and the so-called Lagrange-Souriau form. For higher-order Lagrangian systems the construction of the corresponding Grassmann manifold is more subtle and was performed in [14]. The idea is to start with the manifold of jets of immersions in the kinematical manifold of the problem. This manifold is usually called the velocity manifold and physically corresponds to parametrised evolutions. There exists a natural action of the so-called differential group on this manifold which physically corresponds to changing the parametrisation. One considers the submanifold of the regular velocities and takes its factorisation to the differential group. This is exactly the Grassmann manifold associated to the kinematical manifold of the problem. It was proved that this is the natural framework to describe a Lagrangian system in this more general non-fibrating case. The main combinatorial difficulty consists in establishing a convenient chart system on this factor manifold.

In this paper we continue the analysis of Lagrangian systems in this framework by constructing the corresponding Lagrange, Euler-Lagrange and Helmholtz-Sonin form. The problem to be solved is that the expressions from the fibrating case are no longer well defined geometrical objects so one must find out proper substitutes for them. The idea is to construct these kind of objects first on the velocity manifold as *bona fide* geometrical forms and impose some homogeneity properties. One discovers that these globally defined objects are inducing *locally* defined expressions on the Grassmann manifold which have convenient transformation properties with respect to a change of charts and formally coincide with the desired expressions of the usual Lagrangian formalism. In this way we will be able to define on the Grassmann manifold the classes (modulo contact forms) of the Lagrange, Euler-Lagrange and Helmholtz-Sonin forms.

The paper is organised as follows. In Section 2 we remind the basic construction of a Grassmann manifold following essentially [14] but also providing some new results. We do that because we will need many formulæ for the next sections. For completeness we will also sketch the proof of the main results. In Section 3 we define the main objects of the Lagrangian formalism in the non-fibrating case. In Section 4 we give new proofs for the construction of the Lagrange-Souriau form in the case of first-order Lagrangian systems and suggests some ways of generalisation of the construction to the general case of higher-order systems.

2 Grassmann Manifolds

2.1 The Manifold of (r, n) -Velocities

Let us consider N , $n \geq 1$ and $r \geq 0$ integers such that $n \leq N$, and let X be a smooth manifold of dimension N describing the kinematical degrees of freedom of a certain physical problem.

We will consider $U \subset \mathbb{R}^n$ a neighbourhood of the point $0 \in \mathbb{R}^n$, $x \in X$ and let $\Gamma_{(0,x)}$ be the set of smooth immersions $\gamma : U \rightarrow X$ such that $\gamma(0) = x$. As usual, we consider on $\Gamma_{(0,x)}$ the relation “ $\gamma \sim \delta$ ” iff there exists a chart (V, ψ) $\psi = (x^A)$, $A = 1, \dots, N$ on X such that the functions $\psi \circ \gamma, \psi \circ \delta : \mathbb{R}^n \rightarrow \mathbb{R}^N$ have the same partial derivatives up to order r in the point 0. One can prove as in [12] that \sim is a (chart independent) equivalence relation. By an (r, n) -velocity at a point $x \in X$ we mean such an equivalence class of the type $\Gamma_{(0,x)}/\sim$. The equivalence class of γ will be denoted by $j_0^r \gamma$. The set of (r, n) -velocities at x is denoted by $T_{(0,x)}^r(\mathbb{R}^n, Y) \equiv \Gamma_{(0,x)}/\sim$.

Further, we denote

$$T_n^r X = \bigcup_{x \in X} T_{(0,x)}^r(\mathbb{R}^n, X),$$

and define surjective mappings $\tau_n^{r,s} : T_n^r X \rightarrow T_n^s X$, where $0 < s \leq r$, by $\tau_n^{r,s}(j_0^r \gamma) = j_0^s \gamma$ and $\tau_n^{r,0} : T_n^r X \rightarrow X$, where $1 \leq r$, by $\tau_n^{r,0}(j_0^r \gamma) = \gamma(0)$.

In the conditions above let (V, ψ) , $\psi = (x^A)$, be a chart on X . Then we define the couple (V_n^r, ψ_n^r) where $V_n^r = (\pi_n^{r,0})^{-1}(V)$, $\psi_n^r = (x^A, x_{j_1}^A, \dots, x_{j_1, j_2, \dots, j_r}^A)$, where $1 \leq j_1 \leq j_2 \leq \dots \leq j_r \leq n$, and

$$x_{j_1, \dots, j_k}^A(j_0^r \gamma) \equiv \left. \frac{\partial^k}{\partial t^{j_1} \dots \partial t^{j_k}} x^A \circ \gamma \right|_0, \quad 0 \leq k \leq r. \quad (2.1)$$

Remark 2.0.1 Let us note that the expressions $x_{j_1, \dots, j_k}^A(j_0^r \gamma)$ are defined for all values $j_1, \dots, j_r \in \{1, \dots, n\}$ but because of the symmetry property

$$x_{j_{P(1)}, \dots, j_{P(k)}}^A(j_0^r \gamma) = x_{j_1, \dots, j_k}^A(j_0^r \gamma) \quad (k = 2, \dots, n) \quad (2.2)$$

for all permutations $P \in \mathcal{P}_k$ of the numbers $1, \dots, k$ we consider only the independent components given by the restrictions $1 \leq j_1 \leq j_2 \leq \dots \leq j_r \leq n$. Taking this into account one can use multi-index notations i.e. $\psi_n^r = (x_j^A)$, $|J| = 0, \dots, r$ where by definition $x_\emptyset^A \equiv x^A$. The same comment is true for the partial derivatives $\frac{\partial}{\partial x_{j_1, \dots, j_k}^A}$.

Then one can prove that the couple (V_n^r, ψ_n^r) is a chart on $T_n^r X$ called the *associated chart* of the chart (V, ψ) . Next, one shows that the set $T_n^r X$ has a smooth structure defined by the system of charts (V_n^r, ψ_n^r) ; moreover $T_n^r X$ is a fibre bundle over X with the canonical projection $\tau_n^{r,0}$. The set $T_n^r Y$ endowed with the smooth structure defined by the associated charts defined above is called the *manifold of (r, n) -velocities* over X .

The equations of the mapping $\tau_n^{r,s} : T_n^r X \rightarrow T_n^s X$ in terms of the associated charts are given by $x_{j_1, \dots, j_k}^A \circ \tau_n^{r,s}(j_0^r \gamma) = x_{j_1, \dots, j_k}^A(j_0^s \gamma)$, where $0 \leq k \leq s$. These mappings are all submersions.

2.2 Formal Derivatives

Like in [3], [2], [12], let us consider in the chart (V_n^r, ψ_n^r) the following differential operators

$$\partial_A^{j_1, \dots, j_k} \equiv \frac{r_1! \dots r_n!}{k!} \frac{\partial}{\partial x_{j_1, \dots, j_k}^A}, \quad j_1, \dots, j_k \in \{1, \dots, n\} \quad (2.3)$$

where r_k is the number of times the index k shows up in the sequence j_1, \dots, j_k .

Then one can prove that the following relation is true:

$$\partial_A^{i_1, \dots, i_k} x_{j_1, \dots, j_l}^B = \begin{cases} \delta_A^B \mathcal{S}_{j_1, \dots, j_k}^+ \delta_{j_1}^{i_1} \dots \delta_{j_k}^{i_k} & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}. \quad (2.4)$$

Here we use the notations from [11], namely $\mathcal{S}_{j_1, \dots, j_k}^+$ is the symmetrization (for the sign $+$) and respectively the antisymmetrization (for the sign $-$) projector operator defined by

$$\mathcal{S}_{j_1, \dots, j_k}^\pm f_{j_1, \dots, j_k} \equiv \frac{1}{k!} \sum_{P \in \mathcal{P}_k} \epsilon_\pm(P) f_{j_{P(1)}, \dots, j_{P(k)}} \quad (2.5)$$

where the sum runs over the permutation group \mathcal{P}_k of the numbers $1, \dots, k$ and

$$\epsilon_+(P) \equiv 1, \quad \epsilon_-(P) \equiv (-1)^{|P|}, \quad \forall P \in \mathcal{P}_k;$$

here $|P|$ is the signature of the permutation P .

The differential operators defined by (2.3) take care of overcounting the indices. More precisely, for any smooth function on V^r , the following formula is true:

$$df = \sum_{k=0}^r (\partial_A^{j_1, \dots, j_k} f) dx_{j_1, \dots, j_k}^A = \sum_{|I| \leq r} (\partial_A^I f) dx_I^A \quad (2.6)$$

where we have also used the convenient multi-index notation.

We define now in the chart (V_n^r, ψ_n^r) the *formal derivatives* by the expressions

$$d_i^r \equiv \sum_{k=0}^{r-1} x_{i, j_1, \dots, j_k}^A \partial_A^{j_1, \dots, j_k} = \sum_{|J| \leq r-1} x_{iJ}^A \partial_A^J. \quad (2.7)$$

The last expression uses the multi-index notation; if I and J are two such multi-indices we mean by IJ the juxtaposition of the two sets I, J .

We note that the preceding formula does not define a vector field on $T_n^r Y$. When no danger of confusion exists we simplify the notation putting simply $d_i = d_i^r$. One can easily verify that the following formulæ follow directly from the definition:

$$d_i x_{j_1, \dots, j_k}^A = \begin{cases} x_{i, j_1, \dots, j_k}^A & \text{if } k \leq r-1 \\ 0 & \text{if } k = r \end{cases}, \quad (2.8)$$

$$[\partial_A^{j_1, \dots, j_k}, d_i] = \mathcal{S}_{j_1, \dots, j_k}^+ \delta_i^{j_1} \partial_A^{j_2, \dots, j_k}, \quad k = 0, \dots, r \quad (2.9)$$

and

$$[d_i, d_j] = 0. \quad (2.10)$$

The formal derivatives can be used to conveniently express the change of charts on the velocity manifold induced by a change of charts on X . Let (V, ψ) and $(\bar{V}, \bar{\psi})$ two charts on X such that $V \cap \bar{V} \neq \emptyset$ and let (V^r, ψ^r) and $(\bar{V}^r, \bar{\psi}^r)$ the corresponding attached charts from $T_n^r X$. The change of charts on X is $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ given by: $F \equiv \bar{\psi} \circ \psi^{-1}$. It is convenient to denote by $F^A : \mathbb{R}^N \rightarrow \mathbb{R}$ the components of F given by $F^A \equiv \bar{x}^A \circ \psi^{-1}$. We now consider the change of charts on $T_n^r X$ given by $F^r \equiv \bar{\psi}^r \circ (\psi^r)^{-1}$. One notes that $V^r \cap \bar{V}^r \neq \emptyset$; we need the explicit formulæ for the components of F^r , namely for the functions

$$F_{j_1, \dots, j_k}^A \equiv \bar{x}_{j_1, \dots, j_k}^A \circ (\psi^r)^{-1}, \quad j_1 \leq j_2 \leq \dots \leq j_k, \quad k = 1, \dots, r$$

defined on the overlap: $V^r \cap \bar{V}^r$. First one notes the following relation:

$$\bar{d}_i = d_i. \quad (2.11)$$

Indeed, one defines for any immersion $\gamma \in \Gamma_{(0,x)}$ the map $j^r \gamma$ from \mathbb{R}^n into $T_n^r X$ given by

$$x_{j_1, \dots, j_k}^A \circ j^r \gamma(t) \equiv \frac{\partial^k x^A \circ \gamma}{\partial t^{j_1} \dots \partial t^{j_k}}(t) \quad 0 \leq k \leq r \quad (2.12)$$

and easily discovers that

$$(j^r \gamma)_{*0} \frac{\partial}{\partial t^i} = d_i = \bar{d}_i. \quad (2.13)$$

Using (2.11) one easily finds out that the functions F_{j_1, \dots, j_k}^A are given recurrently by the following relation:

$$F_{jI}^A = d_j F_I^A \quad |I| \leq r-1; \quad (2.14)$$

(compare with (2.8).)

This relation can be “solved” explicitly according to

Lemma 2.1 *The following formula holds*

$$F_I^A = \sum_{p=1}^{|I|} \sum_{(I_1, \dots, I_p)} x_{I_1}^{B_1} \dots x_{I_p}^{B_p} (\partial_{B_1} \dots \partial_{B_p} F^A), \quad 1 \leq |I| \leq r \quad (2.15)$$

where the second sum denotes summation over all partitions $\mathcal{P}(I)$ of the set I and two partitions are considered identical if they differ only by a permutation of the subsets.

Proof: We sketch the proof because the argument will be used repeatedly in this paper. It is natural to use complete induction on $|I|$. For $I = \{j\}$ the formula from the statement coincides with (2.14) for $I = \emptyset$. We suppose the formula true for any multi-index I with $|I| = s < r$ and prove it for the multi-index jI . If we use (2.14) we get:

$$\begin{aligned} F_{jI}^A &= \sum_{p=1}^{|I|} \sum_{(I_1, \dots, I_p)} \left[\sum_{l=1}^p x_{I_1}^{B_1} \dots (d_j x_{I_l}^{B_l}) \dots x_{I_p}^{B_p} (\partial_{B_1} \dots \partial_{B_p} F^A) + x_{I_1}^{B_1} \dots x_{I_p}^{B_p} d_j (\partial_{B_1} \dots \partial_{B_p} F^A) \right] \\ &= \sum_{p=1}^{|I|} \sum_{(I_1, \dots, I_p)} \left[\sum_{l=1}^p x_{I_1}^{B_1} \dots x_{jI_l}^{B_l} \dots x_{I_p}^{B_p} (\partial_{B_1} \dots \partial_{B_p} F^A) + x_{I_1}^{B_1} \dots x_{I_p}^{B_p} x_j^{B_{p+1}} (\partial_{B_1} \dots \partial_{B_{p+1}} F^A) \right]. \end{aligned}$$

We now note that the partitions $\mathcal{P}(jI)$ of the set jI can be obtained in two distinct ways:

- by taking a partition $(I_1, \dots, I_p) \in \mathcal{P}(I)$ and adjoining the index j to I_1, I_2, \dots, I_p ;
- by taking a partition $(I_1, \dots, I_p) \in \mathcal{P}(I)$ and constructing the associated partition $(I_1, \dots, I_p, j) \in \mathcal{P}(jI)$.

We get the two types of contributions in the formula above and this finishes the proof. ■

Remark 2.1.1 *The combinatorial argument above will be called the partition argument.*

Remark 2.1.2 *From the formula derived above it immediately follows that we have:*

$$\partial_B^J F_I^A = 0, \quad 0 \leq |I| < |J| \leq r \quad (2.16)$$

i.e. the functions F_I^A depend only of the variables x_J^B with the restrictions specified above.

2.3 The Differential Group

By definition the *differential group* of order r is the set

$$L_n^r \equiv \{j_0^r \alpha \in J_{0,0}^r(\mathbb{R}^n, \mathbb{R}^n) | \alpha \in \text{Diff}(\mathbb{R}^n)\} \quad (2.17)$$

i.e. the group of invertible r -jets with source and target at $0 \in \mathbb{R}^n$. The group multiplication in L_n^r is defined by the jet composition $L_n^r \times L_n^r \ni (j_0^r \alpha, j_0^r \beta) \mapsto j_0^r(\alpha \circ \beta) \in L_n^r$.

The *canonical* (global) *coordinates* on L_n^r are defined by

$$a_{j_1, \dots, j_k}^i(j_0^r \alpha) = \frac{\partial^k \alpha^i}{\partial t^{j_1} \dots \partial t^{j_k}} \Big|_0, \quad j_1 \leq j_2 \leq \dots \leq j_k, \quad k = 0, \dots, r \quad (2.18)$$

where α^i are the components of a representative α of $j_0^r \alpha$.

We denote

$$a \equiv (a_j^i, a_{j_1, j_2}^i, \dots, a_{j_1, \dots, j_k}^i) = (a_J^i)_{|J| \leq r}$$

and notice that one has

$$\det(a_j^i) \neq 0. \quad (2.19)$$

To obtain the composition law for the differential group we need a combinatorial result following easily by induction with the partition argument:

Lemma 2.2 *Let $U, V \in \mathbb{R}^n$ be open sets, $\alpha : U \rightarrow V$ and $f : V \rightarrow \mathbb{R}$ smooth functions. Then the following formula is true:*

$$\partial_I(f \circ \alpha) = \sum_{p=1}^{|I|} \sum_{(I_1, \dots, I_p)} (\partial_{I_1} \alpha^{i_1}) \dots (\partial_{I_p} \alpha^{i_p}) (\partial_{i_1, \dots, i_p} f) \circ \alpha \quad (2.20)$$

where we have denoted for any multi-index $I = \{i_1, \dots, i_s\}$

$$\partial_I f \equiv \frac{\partial^s f}{\partial t^{i_1} \dots \partial t^{i_s}}.$$

We now have:

Lemma 2.3 *The group multiplication in L_n^r is expressed in the canonical coordinates by the equations*

$$(a \cdot b)_I^k = \sum_{p=1}^{|I|} \sum_{(I_1, \dots, I_p)} b_{I_1}^{j_1} \dots b_{I_p}^{j_p} a_{j_1, \dots, j_p}^k, \quad |I| = 1, \dots, r. \quad (2.21)$$

The group L_n^r is a Lie group.

Proof: (i) We start from the defining formula:

$$(a \cdot b)_{j_1, \dots, j_l}^k = \left. \frac{\partial^l \alpha^k \circ \beta}{\partial t^{j_1} \dots \partial t^{j_l}} \right|_0$$

and apply the lemma above. One obtains the composition formula.

(ii) It is clear that the composition formula (2.21) is a smooth function. The identity is evidently:

$$e \equiv (\delta_j^i, 0, \dots, 0)$$

and it remains to prove that the map $a \rightarrow a^{-1}$ is smooth. Indeed one immediately proves by induction that

$$(a^{-1})_I^k = \left(\det(a_j^i) \right)^{-|I|} \times P_I^k(a)$$

where P_I^k is a polynomial in the variables a_I^i , $|I| = 0, \dots, r$. ■

The manifolds of (r, n) -velocities $T_n^r Y$ admits a (natural) smooth right action of the differential group L_n^r , defined by the jet composition

$$(x \cdot a)_I^A \equiv x_I^A (j_0^r(\gamma \circ \alpha)) \quad (2.22)$$

where the connection between x_I^A and γ is given by (2.1) and the connection between a_I^i and α is given by (2.18).

We determine the chart expression of this action.

Proposition 2.4 *The group action (2.22) is expressed by the equations*

$$(x \cdot a)^A = x^A, \quad (x \cdot a)_I^A = \sum_{p=1}^{|I|} \sum_{(I_1, \dots, I_p) \in \mathcal{P}(I)} a_{I_1}^{j_1} \dots a_{I_p}^{j_p} x_{j_1, \dots, j_p}^A, \quad |I| \geq 1 \quad (2.23)$$

and it is smooth.

Proof:

One applies the definitions (2.1) and (2.18) together with the lemma 2.2. The smoothness is obvious from the explicit action formula given above. ■

The group L_n^r has a natural smooth left action on the set of smooth real functions defined on $T_n^r X$, namely for any such function f we have:

$$(a \cdot f)(x) \equiv f(x \cdot a). \quad (2.24)$$

2.4 Higher Order Regular Velocities

We say that a (r, n) -velocity $j_0^r \gamma \in T_n^r X$ is *regular*, if γ (or any other representative) is an immersion. If (V, ψ) , $\psi = (x^A)$, is a chart, and the target $\gamma(0)$ of an element $j_0^r \gamma \in T_n^r X$ belongs to V , then $j_0^r \gamma$ is regular *iff* there exists a subsequence $\mathbf{I} \equiv (i_1, \dots, i_n)$ of the sequence $(1, 2, \dots, n, n+1, \dots, n+m)$ such that

$$\det(x_j^{i_k}) \neq 0; \quad (2.25)$$

(here $x_j^{i_k}$ is a $n \times n$ real matrix.)

The associated charts have the form

$$(V^{\mathbf{I}^r}, \psi^{\mathbf{I}^r}), \quad \psi^{\mathbf{I}^r} = (x_I^k, x_I^\sigma), \quad k = 1, \dots, n, \quad \sigma = 1, \dots, m \equiv N - n, \quad |I| \leq r$$

where

$$x_I^k \equiv x_I^{i_k}, \quad k = 1, \dots, n$$

and $\sigma \in \{1, \dots, N\} - \{i_1, \dots, i_n\}$. The set of regular (r, n) -velocities is an open, L_n^r -invariant subset of $T_n^r X$, which is called the *manifold of regular (r, n) -velocities*, and is denoted by $\text{Imm}T_n^r X$.

We want to find out a complete system of L_n^r -invariants (in the sense of Weyl) of the action (2.23) on $\text{Imm}T_n^r X$.

We will consider, for simplicity a chart for which one has $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ and we will denote

$$x_I^\sigma \equiv x_I^{n+\sigma}, \quad \sigma = 1, \dots, m, \quad |I| \leq r.$$

We begin with the following result:

Proposition 2.5 *Let (x_I^σ, x_I^i) be the coordinates of a point in $\text{Imm}T_n^r X$. Then*

$$\mathbf{x} \equiv (x_I^i)_{1 \leq |I| \leq r} \quad (2.26)$$

is a element from L_n^r . We denote its inverse by

$$\mathbf{z} \equiv (z_I^i)_{1 \leq |I| \leq r}. \quad (2.27)$$

Then z_j^i is the inverse of the matrix x_p^l :

$$z_j^i x_p^j = \delta_p^i \quad (2.28)$$

and the functions $z_{j_1, \dots, j_k}^i, k = 2, \dots, r$ can be determined recurringly from the equations:

$$z_{j_1, \dots, j_k}^i = z_{j_1}^p d_p z_{j_2, \dots, j_k}^i, \quad k = 2, \dots, r \quad (2.29)$$

Proof: For the first assertion one uses (2.19) and (2.25). For the relation (2.29) one starts from the definition $\mathbf{z} \cdot \mathbf{x} = e$ or, in detail

$$\sum_{k=1}^{|I|} \sum_{(I_1, \dots, I_k)} x_{I_1}^{j_1} \dots x_{I_k}^{j_k} z_{j_1, \dots, j_k}^i = \begin{cases} \delta_I^i & \text{for } |I| = 1 \\ 0 & \text{for } |I| = 2, \dots, r \end{cases}.$$

One performs two distinct operations on this relation: (a) we apply the operator d_p ; (b) we make $I \mapsto Ip$. Next one subtracts the two results and uses the partition argument; the following relation follows:

$$\sum_{k=1}^{|I|} \sum_{(I_1, \dots, I_k)} x_{I_1}^{j_1} \dots x_{I_k}^{j_k} (d_p z_{j_1, \dots, j_k}^i - x_p^{j_0} z_{j_0, \dots, j_k}^i) = 0.$$

From this relation, we obtain, by induction the formula from the statement. ■

The formula (2.29) suggests the following result:

Proposition 2.6 *Let $(V, \psi), \psi = (x^A)$, be a chart on X and let (V_n^r, ψ_n^r) be the associated chart on $\text{Imm}T_n^r X$. We define recurringly on this chart the following functions*

$$y^\sigma \equiv x^\sigma, \quad y_{i_1, \dots, i_k}^\sigma = z_{i_1}^j d_j y_{i_2, \dots, i_k}^\sigma, \quad k = 1, \dots, r; \quad (2.30)$$

(here z_i^j are the first entries of the element $\mathbf{z} \in L_n^r$.)

Then the functions $y_{i_1, \dots, i_k}^\sigma$ so defined depend smoothly only on x_J^A , $|J| \leq k$ and are completely symmetric in all indices i_1, \dots, i_k , $k = 1, \dots, r$.

Proof:

The first assertion follows immediately by induction. Next, one derives directly from the formula (2.30) that

$$y_{i_1, \dots, i_k}^\sigma = z_{i_1}^{j_1} z_{i_2}^{j_2} \left(d_{j_1} d_{j_2} y_{i_3, \dots, i_k}^\sigma - x_{i_1, i_2}^p z_p^{j_1} d_{j_1} y_{i_3, \dots, i_k}^\sigma \right), \quad k = 2, \dots, r.$$

In particular we see that for $k = 2$ the symmetry property is true. One can proceed now by induction. If $y_{i_1, \dots, i_{k-1}}^\sigma$ is completely symmetric then the formula above shows that we have the symmetry property in the indices i_1 and i_2 ; moreover the recurrence relation (2.30) shows that we have the symmetry property in the indices i_2, \dots, i_k . So we obtain the desired property in all indices. ■

As a result of the symmetry property just proved we can use the convenient multi-index notation y_I^σ , $|I| \leq r$. Now we have an explicit formula for these functions

Proposition 2.7 *The functions y_I^σ , $1 \leq |I| \leq r$ are **uniquely** determined by the recurrence relations:*

$$x_I^\sigma = \sum_{p=1}^{|I|} \sum_{(I_1, \dots, I_p)} x_{I_1}^{j_1} \dots x_{I_p}^{j_p} y_{j_1, \dots, j_p}^\sigma. \quad (2.31)$$

Using the notation $\mathbf{x} \in L_n^r$ one can compactly write the relation above as

$$x_I^\sigma = (y \cdot \mathbf{x})_I^\sigma, \quad 1 \leq |I| \leq r. \quad (2.32)$$

Proof: Goes by induction on $|I|$. The formula above is obvious for $I = \{j\}$. If it is valid for $|I| < r$ we apply to the relation above the operator d_j and use (2.8) and the partition argument. One obtains in this way the formula from the statement for Ij . The unicity also follows by induction. The last assertion is a consequence of the first formula and of the expression of the group action (2.23). ■

Let us note that one can “invert” the formulæ from the statement. Indeed, (2.32) is equivalent to

$$y_I^\sigma = (x \cdot \mathbf{z})_I^\sigma, \quad 1 \leq |I| \leq r \quad (2.33)$$

or, explicitly:

$$y_I^\sigma = \sum_{p=1}^{|I|} \sum_{(I_1, \dots, I_p)} z_{I_1}^{j_1} \dots z_{I_p}^{j_p} x_{j_1, \dots, j_p}^\sigma. \quad (2.34)$$

Corollary 2.8 *One can use on V^r the new coordinates (y_I^σ, x_I^i) , $|I| \leq r$.*

Proof: From the relations (2.31) and (2.34). ■

Now we have the following result

Proposition 2.9 *The functions y_I^σ , $|I| \leq r$ are L_n^r -invariants with respect to the natural action (2.24).*

Proof: Let $a \in L_n^r$ be arbitrary. We start from (2.32) and use the associativity of the group composition law of L_n^r ; we get:

$$(x \cdot a)_I^\sigma = ((y \cdot \mathbf{x}) \cdot a)_I^\sigma = (y \cdot (\mathbf{x} \cdot a))_I^\sigma.$$

On the other hand if we make in (2.32) the substitution $x \mapsto x \cdot a$ we get:

$$(x \cdot a)_I^\sigma = ((a \cdot y) \cdot (x \cdot a))_I^\sigma.$$

(here $a \cdot y$ denotes the action of the differential group on the functions y according to (2.24).)

By comparing the two formulæ and using of the unicity statement from the preceding proposition we get the desired result. ■

Moreover, we can prove:

Theorem 2.10 *The functions y_I^σ , $|I| \leq r$ are a complete system of invariants in the sense of Weyl.*

Proof: The fact that the functions y_I^σ , $|I| \leq r$ are functionally independent follows by *reductio ad absurdum*. If they would be functionally dependent, then from (2.34) it would follow that the expressions x_I^σ , $|I| \leq r$ also are functionally dependent.

We still must show that there are no other invariants beside y_I^σ , $|I| \leq r$. We proceed as follows. From corollary 2.8 it follows that one can use on V^r the coordinates (y_I^σ, x_I^i) , $|I| \leq r$. In these coordinates the action of the group L_n^r is:

$$(y \cdot a)_I^\sigma = y_I^\sigma, \quad |I| \leq r, \\ (x \cdot a)^i = x^i, \quad (x \cdot a)_I^i = \sum_{p=1}^{|I|} \sum_{(I_1, \dots, I_p)} a_{I_1}^{j_1} \dots a_{I_p}^{j_p} x_{j_1, \dots, j_p}^i, \quad 1 \leq |I| \leq r. \quad (2.35)$$

One can prove now by induction that this action is transitive. This shows that the system of invariants from the statement is complete. ■

2.5 Higher Order Grassmann Bundles

Needless to say, the whole formalism presented above can be implemented in an arbitrary chart system $(V^{\mathbf{I},r}, \psi^{\mathbf{I},r})$ on $\text{Imm}T_n^r X$ (see the beginning of the preceding subsection). In this context we finally have the central result:

Theorem 2.11 *The set $P_n^r X \equiv \text{Imm}T_n^r X / L_n^r$ has a unique differential manifold structure such that the canonical projection ρ_n^r is a submersion. The group action (2.23) defines on $\text{Imm}T_n^r X$ the structure of a right principal L_n^r -bundle.*

A chart system on $P_n^r X$ adapted to this fibre bundle structure is formed from couples $(W^{\mathbf{I},r}, \Phi^{\mathbf{I},r})$ where:

$$W^{\mathbf{I},r} = \{j_0^r \gamma \in V^r | \det(x_j^{i_k}(j_0^r \gamma)) \neq 0\} \quad (2.36)$$

and

$$\Phi^{\mathbf{I},r} = (x_I^i, y_I^\sigma), \quad |I| \leq r. \quad (2.37)$$

In this case the local expression of the canonical projection is

$$\rho_n^r(x_I^i, y_I^\sigma) = (x^i, y_I^\sigma).$$

Proof: We define on $\text{Imm}T_n^r X \times \text{Imm}T_n^r X$ the equivalence relation

$$x \sim \bar{x} \quad \text{iff} \quad \exists a \in L_n^r \quad \text{s.t.} \quad \bar{x} = x \cdot a.$$

To prove the first assertion from the statement is sufficient (according to [7], par. 16.19.3) to prove that the graph of \sim is a closed submanifold of the product manifold. We will look for a convenient system of coordinates on $\text{Imm}T_n^r X$.

The first step is to take x and \bar{x} such that $x \sim \bar{x}$ and to solve the system of equations

$$\bar{x} = x \cdot a_{x,\bar{x}}$$

for the unknown functions $a_{x,\bar{x}} \in L_n^r$.

One easily gets

$$(a_{x,\bar{x}})_j^i = z_p^i \bar{x}_j^p$$

and then shows by induction that $(a_{x,\bar{x}})_I^i$ are uniquely determined smooth functions of x_J^i, \bar{x}_J^i , $|J| \leq |I| \leq r$.

We now define the (local) smooth functions on $\text{Imm}T_n^r X \times \text{Imm}T_n^r X$:

$$\Phi_I^\sigma(x, \bar{x}) \equiv \bar{x}_I^\sigma - (x \cdot a_{x,\bar{x}})_I^\sigma, \quad |I| \leq r.$$

It is clear that on can take on $\text{Imm}T_n^r X \times \text{Imm}T_n^r X$ the (local) coordinates $(x_I^A, \Phi_I^\sigma, \bar{x}_I^i)$, $|I| \leq r$; it follows that the graph of \sim is given by

$$\Phi_I^\sigma = 0, \quad |I| \leq r$$

i.e. it is a closed submanifold.

To prove the fibre bundle structure it is sufficient to show (see also [7]) that the action of L_n^r is free i.e.

$$x \cdot a = x \implies a = e.$$

This fact follows elementary by induction.

Finally, we have remarked before (see the preceding theorem) that one can take on $\text{Imm}T_n^r X$ the coordinates (y_I^σ, x_I^i) , $|I| \leq r$ with the action given by (2.35). The last assertion about the expression of the canonical projection follows. ■

A point of $P_n^r X$ containing a regular (r, n) -velocity $j_0^r \gamma$ is called an (r, n) -*contact element*, or an r -*contact element* of an n -dimensional submanifold of X , and is denoted by $[j_0^r \gamma]$. As in the case of r -jets, the point $0 \in \mathbb{R}^n$ (resp. $\gamma(0) \in X$) is called the *source* (resp. the *target*) of $[j_0^r \gamma]$. The manifold P_n^r is called the (r, n) -*Grassmannian bundle*, or simply a *higher order Grassmannian bundle* over X .

Besides the quotient projection $\rho_n^r : \text{Imm}T_n^r X \rightarrow P_n^r$ we have for every $1 \leq s \leq r$, the *canonical projection* of $P_n^r X$ onto $P_n^s X$ defined by $\rho_n^{r,s}([j_0^r \gamma]) = [j_0^s \gamma]$ and the *canonical projection* of $P_n^r X$ onto X defined by $\rho_n^r([j_0^r \gamma]) = \gamma(0)$.

Remark 2.11.1 When the manifold X is fibred over a manifold M of dimension n one can also construct the jet extension $J^r X$ (see [21], [12]). One can establish a canonical isomorphism between $P_n^s X$ and $J^r X$ as follows: let $x \in M, x = \gamma(m)$, $\gamma \in \Gamma_{(m,x)}$; and $\phi : \mathbb{R}^n \rightarrow M$ a (local) diffeomorphism such that $\phi(0) = m$. We can define $\tilde{\gamma} \in \Gamma_{(0,x)}$ by the formula $\tilde{\gamma} \equiv \gamma \circ \phi$. One notices that $\tilde{\gamma}_1 \sim \tilde{\gamma}_2$ iff there exists $\alpha \in \text{Diff}(\mathbb{R}^n)$ such that $\gamma_1 = \gamma_2 \circ \alpha$. This means that the map $j_n^r \gamma \mapsto j^r \tilde{\gamma}$ can be factorized to a map from $P_n^r X \rightarrow J^r X$ which is proved to be an isomorphism.

We also note the following result:

Proposition 2.12 The following formula is true

$$\Phi_I^\sigma = \sum_{p=1}^{|I|} \sum_{(I_1, \dots, I_p)} \bar{x}_{I_1}^{j_1} \dots \bar{x}_{I_p}^{j_p} (\bar{y}_{j_1, \dots, j_p}^\sigma - y_{j_1, \dots, j_p}^\sigma), \quad 1 \leq |I| \leq r. \quad (2.38)$$

or, in compact notations

$$\Phi_I^\sigma = ((\bar{y} - y) \cdot \bar{x})_I^\sigma. \quad (2.39)$$

In particular, the equation

$$\Phi_I^\sigma = 0, \quad 1 \leq |I| \leq r$$

is equivalent to

$$\bar{y}_I^\sigma = y_I^\sigma, \quad |I| \leq r.$$

Proof: The proof relies heavily on induction. Firstly, we define on $\text{Imm} T_n^r X \times \text{Imm} T_n^r X$ the expressions

$$V_i \equiv \bar{d}_i - (a_{x, \bar{x}})_i^j d_j$$

(where $a_{x, \bar{x}}$ have been defined previously) and we prove by induction the following formula:

$$V_i(a_{x, \bar{x}})_I^j = (a_{x, \bar{x}})_i^j, \quad 1 \leq |I| \leq r-1.$$

Next, one uses this formula to prove by direct computation that

$$V_i \Phi_I^\sigma = \Phi_{iI}^\sigma.$$

Finally, one uses the preceding formula to prove by induction the formula (2.38) from the statement. ■

Remark 2.12.1 The dimension of the factor manifold $P_n^r X$ is

$$\dim P_n^r X = m \binom{n+r}{n} + n.$$

Now we try to define on $P_n^r X$ the analogue of the total differential operators.

Proposition 2.13 *Let us consider on the regular velocities manifold $\text{Imm}T_n^r X$ the coordinates (y_I^σ, x_I^i) , $|I| \leq r$ and define the operators*

$$\tilde{\Delta}_\sigma^{j_1, \dots, j_k} \equiv \frac{r_1! \dots r_n!}{k!} \frac{\partial}{\partial y_{j_1, \dots, j_k}^\sigma} \quad (2.40)$$

(where we use the same conventions as in (2.3)).

We also define, by analogy to (2.7)

$$\tilde{d}_i^r \equiv \frac{\partial}{\partial x^i} + \sum_{k=0}^{r-1} y_{i, j_1, \dots, j_k}^\sigma \tilde{\Delta}_\sigma^{j_1, \dots, j_k} = \sum_{|J| \leq r-1} y_{iJ}^\sigma \tilde{\Delta}_\sigma^J. \quad (2.41)$$

Then the following formula is true

$$d_i = \sum_{p=1}^{r-1} x_{i, j_1, \dots, j_k}^l \partial_l^{j_1, \dots, j_k} + x_i^p \tilde{d}_p \quad (2.42)$$

where, as usual, $\tilde{d}_i = \tilde{d}_i^r$ when no danger of confusion arises.

Proof: By direct computation. ■

Now we define on $P_n^r X$, in the chart $\rho_n^r(W^{\mathbf{I}, r})$ some operators which are the analogues of (2.40) and (2.7), namely

$$\Delta_\sigma^{j_1, \dots, j_k} \equiv \frac{r_1! \dots r_n!}{k!} \frac{\partial}{\partial y_{j_1, \dots, j_k}^\sigma} \quad (2.43)$$

and

$$D_i \equiv \frac{\partial}{\partial x^i} + \sum_{k=0}^{r-1} y_{i, j_1, \dots, j_k}^\sigma \Delta_\sigma^{j_1, \dots, j_k} = \frac{\partial}{\partial x^i} + \sum_{|J| \leq r-1} y_{iJ}^\sigma \Delta_\sigma^J. \quad (2.44)$$

These operators are also called *total derivatives*. A formula similar to (2.4) is valid and moreover, the preceding proposition has the following consequence:

Proposition 2.14 *The following formula is true:*

$$(\rho_n^r)_*(z_i^j d_j) = D_i. \quad (2.45)$$

In particular, we have for any smooth function f on $\rho_n^r(W^r)$ the following formula:

$$d_i(f \circ \rho_n^r) = x_i^j (D_j f) \circ \rho_n^r. \quad (2.46)$$

Therefore, if (V, ψ) and $(\bar{V}, \bar{\psi})$ are two charts on X such that $V \cap \bar{V} \neq \emptyset$ and D_i, \bar{D}_i , $i = 1, \dots, n$ are the corresponding operators defined on $\rho_n^r(V^r)$ and respectively on $\rho_n^r(\bar{V}^r)$, then we have on $\rho_n^r(V^r \cap \bar{V}^r)$:

$$\text{Span}(D_1, \dots, D_n) = \text{Span}(\bar{D}_1, \dots, \bar{D}_n). \quad (2.47)$$

Proof: The first formula follows directly from the preceding proposition. For the second formula one also applies (2.11). ■

Finally we can give the formula for the chart change on $P_n^r X$.

Proposition 2.15 *In the conditions of the preceding proposition, let $(\rho_n^r(V^r), (x^i, y^\sigma))$ and respectively $(\rho_n^r(\bar{V}^r), (\bar{x}^i, \bar{y}^\sigma))$ be the two (overlapping charts); then the change of charts on $\rho_n^r(V^r) \cap \rho(\bar{V}^r)$ is given by:*

$$\bar{y}_{iI}^\sigma = P_i^j D_j \bar{y}_I^\sigma, \quad |I| \leq r-1 \quad (2.48)$$

where P is the inverse of the matrix Q :

$$Q_p^l \equiv D_p \bar{x}^l, \quad P_i^j Q_j^l = \delta_i^l. \quad (2.49)$$

Proof:

We have from (2.30)

$$\bar{x}_j^i \bar{y}_{iI}^\sigma = \bar{d}_j \bar{y}_I^\sigma, \quad |I| \leq r-1$$

with \bar{y}_I^σ functions of \bar{x}_j^A .

We will consider this relation on the overlap $V^r \cap \bar{V}^r$ such that \bar{y}_I^σ can be considered as functions of x_j^A through the chart transformation formulæ. Using also (2.11) one gets:

$$\bar{x}_j^i \bar{y}_{iI}^\sigma = d_j \bar{y}_I^\sigma. \quad |I| \leq r-1$$

We rewrite this relation in the new coordinates (x^i, y_I^σ, x_I^i) (see corollary 2.8) and also use (2.42); as a result one finds out:

$$z_p^j \bar{x}_j^i \bar{y}_{iI}^\sigma = \tilde{d}_p \bar{y}_I^\sigma, \quad |I| \leq r-1.$$

It remains to prove using also (2.42) that

$$Q_p^j = z_p^j \bar{x}_j^i \quad (2.50)$$

and the change transformation formula from the statement follows. ■

We now note two other properties of the total differential operators D_i . The first one follows immediately from (2.45) and (2.50):

$$Q_i^j \bar{D}_j = D_i. \quad (2.51)$$

The second one is the analogue of (2.10):

$$[D_i, D_j] = 0. \quad (2.52)$$

So the expression

$$D_I \equiv \prod_{i \in I} D_i$$

makes sense for every multi-index I .

We close this subsection with a result which will be useful later.

Proposition 2.16 *The following formula is true on the overlap of two charts:*

$$\Delta_{\sigma}^{i_1, \dots, i_k} \bar{y}_{j_1, \dots, j_k}^{\nu} = \mathcal{S}_{j_1, \dots, j_k}^{+} P_{j_1}^{i_1} \dots P_{j_k}^{i_k} P_{\sigma}^{\nu}, \quad k = 1, \dots, r. \quad (2.53)$$

Proof: It is done by recurrence. First one proves directly from the definitions that:

$$\Delta_{\sigma}^{i_1, \dots, i_k} \bar{y}_{j_1, \dots, j_k}^{\nu} = \mathcal{S}_{j_1, \dots, j_k}^{+} P_{j_1}^{i_1} \Delta_{\sigma}^{i_2, \dots, i_k} \bar{y}_{j_2, \dots, j_k}^{\nu}, \quad k = 2, \dots, r \quad (2.54)$$

and then we obtain by recurrence:

$$\Delta_{\sigma}^{i_1, \dots, i_k} \bar{y}_{j_1, \dots, j_k}^{\nu} = \mathcal{S}_{j_1, \dots, j_k}^{+} P_{j_1}^{i_1} \dots P_{j_{k-1}}^{i_{k-1}} \Delta_{\sigma}^{i_k} \bar{y}_{j_k}^{\nu}, \quad k = 1, \dots, r. \quad (2.55)$$

Finally one establishes by direct computation that

$$\Delta_{\sigma}^i \bar{y}_j^{\nu} = P_j^i P_{\sigma}^{\nu} \quad (2.56)$$

and the formula from the statement follows. ■

As a corollary we have the following fact:

Corollary 2.17 *Let us denote by $\Omega_q^r(PX)$, $q \geq 0$ the modulus of differential forms of order q on P_n^r . Then the subspace*

$$\Omega_{q, hor}^r(PX) \equiv \{\alpha \in \Omega_q^r(PX) | i_{\Delta_{\sigma}^{j_1, \dots, j_r}} \alpha = 0\}$$

is globally well defined.

Proof: One has, according to the chain rule on the overlap of two charts:

$$\Delta_{\sigma}^{i_1, \dots, i_r} = \left(\Delta_{\sigma}^{i_1, \dots, i_r} \bar{y}_{j_1, \dots, j_r}^{\nu} \right) \bar{\Delta}_{\nu}^{i_1, \dots, i_r} = P_{j_1}^{i_1} \dots P_{j_r}^{i_r} P_{\sigma}^{\nu} \bar{\Delta}_{\nu}^{i_1, \dots, i_r}$$

and a similar formula for the corresponding inner contractions. It follows that the relation

$$i_{\Delta_{\sigma}^{j_1, \dots, j_r}} \alpha = 0$$

is chart independent. ■

2.6 Contact Forms on Grassmann Manifolds

In this subsection we give some new material about the possibility of defining the contact forms on the factor manifold $P_n^r X$. Fortunately, most of the definitions and properties from [21]-[23] and [12] can be adapted to this more general situation.

By a *contact form* on $P_n^r X$ we mean any form $\rho \in \Omega_q^r(PX)$ verifying

$$[j^r \gamma]^* \rho = 0 \quad (2.57)$$

for any immersion $\gamma : \mathbb{R}^n \rightarrow X$. We denote by $\Omega_{q(c)}^r(PX)$ the set of contact forms of degree $q \leq n$. Here $[j^r \gamma] : \mathbb{R}^n \rightarrow P_n^r X$ is given by (see def. (2.12)) $[j^r \gamma](t) \equiv [j_t^r \gamma]$. Now, many results from [12] are practically unchanged. We mention some of them.

If one considers only the contact forms on an open set $\rho_n^r(V^r) \subset P_n^r X$ then we emphasize this by writing $\Omega_{q(c)}^r(V)$. One immediately notes that $\Omega_{0(c)}^r = 0$ and that for $q > n$ any q -form is contact. It is also elementary to see that the set of all contact forms is an ideal, denoted by $\mathcal{C}(\Omega^r)$, with respect to the operation \wedge . Because the operations of pull-back and of differentiation are commuting this ideal is left invariant by exterior differentiation:

$$d\mathcal{C}(\Omega^r) \subset \mathcal{C}(\Omega^r). \quad (2.58)$$

By elementary computations one finds out that, as in the case of a fibre bundle, for any chart (V, ψ) on X , every element of the set $\Omega_{1(c)}^r(V)$ is a linear combination of the following expressions:

$$\omega_{j_1, \dots, j_k}^\sigma \equiv dy_{j_1, \dots, j_k}^\sigma - y_{i, j_1, \dots, j_k}^\sigma dx^i, \quad k = 0, \dots, r-1 \quad (2.59)$$

or, in multi-index notations

$$\omega_J^\sigma \equiv dy_J^\sigma - y_{iJ}^\sigma dx^i, \quad |J| \leq r-1. \quad (2.60)$$

From the definition above it is clear that the linear subspace of the 1-forms on P_n^r is generated by dx^i , ω_J^σ , ($|J| \leq r-1$) and dy_I^σ , $|I| = r$.

For any smooth function on $\rho(V^r)$ we have

$$df = (d_i f) dx^i + \sum_{|J| \leq r-1} (\partial_\sigma^J f) \omega_J^\sigma + \sum_{|I|=r} (\partial_\nu^I f) dy_I^\nu. \quad (2.61)$$

We also have the formula

$$d\omega_J^\sigma = -\omega_{iJ}^\sigma \wedge dx^i, \quad |J| \leq r-2. \quad (2.62)$$

The structure theorem from [23], [12] stays true, i.e. any $\rho \in \Omega_q^r(PX)$, $q = 2, \dots, n$ is contact *iff* it has the following expression in the associated chart:

$$\rho = \sum_{|J| \leq r-1} \omega_J^\sigma \wedge \Phi_\sigma^J + \sum_{|I|=r-1} d\omega_I^\sigma \wedge \Psi_\sigma^I \quad (2.63)$$

where $\Phi_\sigma^J \in \Omega_{q-1}^r$ and $\Psi_\sigma^I \in \Omega_{q-2}^r$ can be arbitrary forms. (We adopt the convention that $\Omega_q^r \equiv 0, \forall q < 0$).

We will need in the following the transformation formula relevant for change of charts. It is to be expected that there will be some modifications of the corresponding formula from the fibre bundle case. Namely, we have:

Proposition 2.18 *Let (V, ψ) and $(\bar{V}, \bar{\psi})$ two overlapping charts on X and let (W^r, Φ^r) , $\Phi^r = (x^i, y_I^\sigma, x_I^i)$ and $(\bar{W}^r, \bar{\Phi}^r)$, $\bar{\Phi}^r = (\bar{x}^i, \bar{y}_I^\sigma, \bar{x}_I^i)$ the corresponding charts on $T_n^r X$. Then the following formula is true on $\rho_n^r(W^r \cap \bar{W}^r)$:*

$$\bar{\omega}_I^\sigma = \sum_{|J| \leq |I|} (\Delta_\nu^J \bar{y}_I^\sigma) \omega_J^\nu - \bar{y}_{jI}^\sigma (\Delta_\nu \bar{x}^j) \omega^\nu, \quad |I| \leq r-1. \quad (2.64)$$

In particular,

$$\bar{\omega}^\sigma = P_\nu^\sigma \omega^\nu \quad (2.65)$$

where we have defined:

$$P_\nu^\sigma \equiv \Delta_\nu \bar{y}^\sigma - \bar{y}_i^\sigma (\Delta_\nu \bar{x}^i). \quad (2.66)$$

The proof goes by elementary computations. As a consequence we have:

Corollary 2.19 *If a q -form has the expression*

$$\rho = \sum_{p+s=q-n+1} \sum_{|J_1|, \dots, |J_p| \leq r-1} \sum_{|I_1|, \dots, |I_s| = r-1} \omega_{J_1}^{\sigma_1} \cdots \wedge \omega_{J_p}^{\sigma_p} \wedge d\omega_{I_1}^{\nu_1} \cdots \wedge d\omega_{I_s}^{\nu_s} \wedge \Phi_{\sigma_1, \dots, \sigma_p, \nu_1, \dots, \nu_s}^{J_1, \dots, J_p, I_1, \dots, I_s} \quad (2.67)$$

is valid in one chart, then it is valid in any other chart.

This corollary allows us to define for any $q = n+1, \dots, N \equiv \dim(J^r Y) = m \binom{n+r}{n}$ a *strongly contact form* to be any $\rho \in \Omega_q^r$ such that it has in one chart (thereafter in any other chart) the expression above. For a certain uniformity of notations, we denote these forms by $\Omega_{q(c)}^r$.

Now it follows that one can define the variational sequence and prove its exactness as in the fibre bundle case.

We also mention the fact that one can define a global operator on the linear space $\Omega_{q,hor}^r X$ defined at the end of the preceding subsection. In fact we have

Proposition 2.20 *Let $r \geq 1$. Then, the operator locally defined on any differential form by:*

$$K\alpha \equiv i_{D_{j_1}} i_{\Delta_\sigma^{j_1, \dots, j_{r-1}}} \left(\omega_{j_2, \dots, j_{r-1}}^\sigma \wedge \alpha \right) \quad (2.68)$$

is globally defined on the subspace $\Omega_{q,hor}^r X$.

Proof: One works on the overlap of two charts and starts from the definition above trying to transform everything into the other set of coordinates. It is quite elementary to use corollary 2.17 to find

$$K\alpha \equiv i_{\bar{D}_{j_1}} i_{\bar{\Delta}_\nu^{j_1, \dots, j_{r-1}}} \left(P_{k_2}^{j_2} \cdots P_{k_{r-1}}^{j_{r-1}} P_\sigma^\nu \omega_{j_2, \dots, j_{r-1}}^\sigma \wedge \alpha \right)$$

Now one uses (2.53) and the transformation formula (2.64) for the contact forms to obtain

$$K\alpha = \bar{K}\bar{\alpha}.$$

that it, K is well defined globally. ■

2.7 Morphisms of Grassmannian Manifolds

Let X_i , $i = 1, 2$ be two differential manifolds and $\phi : X_1 \rightarrow X_2$ a smooth map. We define the new map $j^r\phi : T_n^r X_1 \rightarrow T_n^r X_2$ according to

$$j^r\phi(j_0^r\gamma) \equiv j_0^r\phi \circ \gamma \quad (2.69)$$

for any immersion γ . If γ is a regular immersion, then one can see that the map $j^r\phi$ maps $\text{Imm}T_n^r(X_1)$ into $\text{Imm}T_n^r(X_2)$ and so, it factorizes to a map $J^r\phi : P_n^r X_1 \rightarrow P_n^r X_2$ given by

$$J^r\phi([j_0^r\gamma]) \equiv [j_0^r\phi \circ \gamma]. \quad (2.70)$$

The map $J^r\phi$ is called the *extension of order r of the map ϕ* .

One can show that the contact ideal behaves naturally with respect to prolongations i.e.

$$(J^r\phi)^*\mathcal{C}(\Omega^r(PX_1)) \subset \mathcal{C}(\Omega^r(PX_2)). \quad (2.71)$$

The proof follows directly from the definition of a contact form.

If ξ is a vector field on X we define its *extension of order r* on $T_n^r X$ and on $P_n^r X$ the vector fields $j^r\xi$ and $J^r\xi$ respectively given by the following formulæ:

$$j^r\xi_{j_0^r\gamma}f \equiv \left. \frac{d}{dt}f \circ j^r e^{t\xi}(j_0^r\gamma) \right|_{t=0} \quad (2.72)$$

(for any smooth real function f on $T_n^r X$) and

$$J^r\xi \equiv \left. \frac{d}{dt}J^r e^{t\xi} \right|_{t=0}; \quad (2.73)$$

here $e^{t\xi}$ is, as usual, the flow associated to ξ .

One will need the explicit formula of $j^r\xi$. If in the chart (V, ψ) we have

$$\xi = \xi^A \partial_A \quad (2.74)$$

with ξ^A smooth function, then $j^r\xi$ has the following expression in the associated chart (V^r, ψ^r) :

$$j^r\xi = \sum_{|I| \leq r} (d_J \xi^A) \partial_A^J. \quad (2.75)$$

The proof of this fact follows by direct computation from the definition above. We call *evolutions* these type of vector fields on $T_n^r X$ and denote the set of evolutions by $\mathcal{E}(T_n^r X)$.

As a consequence of (2.71), if ξ is a vector field on X , then

$$L_{J^r \xi} \mathcal{C}(\Omega^r(X)) \subset \mathcal{C}(\Omega^r(X)). \quad (2.76)$$

Now, as in [12] we have the following results. Suppose that in local coordinates we have

$$\xi = a^i(x, y) \frac{\partial}{\partial x^i} + b^\sigma(x, y) \Delta_\sigma \quad (2.77)$$

with a^i and b^σ smooth function; then $J^r \xi$ must have the following expression in the associated chart:

$$J^r \xi = a^i(x) \frac{\partial}{\partial x^i} + \sum_{|J| \leq r} b_J^\sigma \Delta_\sigma^J. \quad (2.78)$$

where

$$b_J^\sigma = D_I(b^\sigma - y_j^\sigma a^j) + y_{jI}^\sigma a^j, \quad |I| \leq r-1, \quad b_I^\sigma = D_I(b^\sigma - y_j^\sigma a^j), \quad |I| = r. \quad (2.79)$$

Finally we give the expression of the prolongation $J^r \phi$ where ϕ is a bundle morphism of the X . If ϕ has the local expression

$$\phi(x^i, y^\sigma) = (f^i, F^\sigma) \quad (2.80)$$

then we must have in the associated chart:

$$J^r \phi(x^i, y^\sigma, y_j^\sigma, \dots, y_{j_1, \dots, j_r}^\sigma) = (f^i, F^\sigma, F_j^\sigma, \dots, F_{j_1, \dots, j_r}^\sigma) \quad (2.81)$$

where $F_{j_1, \dots, j_k}^\sigma$, $j_1 \leq j_2 \leq \dots \leq j_k$, $k = 1, \dots, r$ are smooth local functions given recurringly by:

$$F_{ji}^\sigma = P_i^l D_l F_j^\sigma \quad |J| \leq r-1; \quad (2.82)$$

we also have

$$\Delta_\nu^I F_J^\sigma = 0 \quad |I| = r. \quad (2.83)$$

3 The Lagrangian Formalism on a Grassmann Manifold

3.1 Euler-Lagrange Forms

We outline a construction from [2] which is the main combinatorial trick in the study of globalisation of the Lagrangian formalism. We call any map $P : \mathcal{E}(T_n^r X) \rightarrow \Omega^s(T_n^s X)$, $s \geq r$ covering the identity map: $id : T_n^r X \rightarrow T_n^r X$ a *total differential operator*. In local coordinates such an operator has the following expression: if ξ has the local expression (2.74), then:

$$P(\xi) = \sum_{|I| \leq r} (d_I \xi^A) P_A^I = \sum_{k=0}^r \left(d_{j_1} \dots d_{j_k} \xi^A \right) P_A^{j_1, \dots, j_k} \quad (3.1)$$

where $P_A^{j_1, \dots, j_k}$ are differential forms on $T_n^s X$ and $d_j = d_j^s$.

Then, as in [2] and [12] one has the following combinatorial lemma:

Lemma 3.1 *In the conditions above, the following formula is true:*

$$P(\xi) = \sum_{|I| \leq r} d_I (\xi^A Q_A^I) \quad (3.2)$$

where

$$Q_A^I \equiv \sum_{|J| \leq r - |I|} (-1)^{|J|} \binom{|I| + |J|}{|I|} d_J P_A^{IJ} \quad (3.3)$$

and one assumes that the action of a formal derivative d_j on a form is realized by its action on the function coefficients. Moreover, the relation (3.2) **uniquely** determines the forms Q_σ^I .

The proof is identical with the one presented in [12]. We also have

Proposition 3.2 *In the conditions above one has on the overlap $V^s \cap \bar{V}^s$ the following formula:*

$$Q_A = (\partial_A \bar{x}^B) \bar{Q}_B. \quad (3.4)$$

In particular, there exists a globally defined form, denoted by $E(P)(\xi)$ with the local expression

$$E(P)(\xi) = Q_A \xi^A. \quad (3.5)$$

Proof: From the formula (3.2) we have

$$P(\xi) = \xi^A Q_A + \sum_{k=1}^r d_{j_1} \dots d_{j_k} \left(\xi^A Q_A^{j_1, \dots, j_k} \right)$$

So, in the overlap $V^s \cap \bar{V}^s$ we have

$$\xi^A Q_A - \bar{\xi}^A \bar{Q}_A = \sum_{k=1}^r \left[\bar{d}_{j_1} \dots \bar{d}_{j_k} \left(\bar{\xi}^A \bar{Q}_A^{j_1, \dots, j_k} \right) - d_{j_1} \dots d_{j_k} \left(\xi^A Q_A^{j_1, \dots, j_k} \right) \right].$$

But because of the relation (2.11) we can simplify considerably this formula, namely we get:

$$\xi^A Q_A - \bar{\xi}^A \bar{Q}_A = \sum_{k=1}^r d_{j_1} \dots d_{j_k} \left(\bar{\xi}^A \bar{Q}_A^{j_1, \dots, j_k} - \xi^A Q_A^{j_1, \dots, j_k} \right)$$

Now one proves that both sides are zero as in [2], [12] making use of Stokes theorem. ■

The operator $E(P)$ defined by (3.5) is called the *Euler operator* associated to the total differential operator P ; it has the local expression:

$$E(P)(\xi) = \xi^A E_A(P) \quad (3.6)$$

where

$$E_A(P) = \sum_{|I|=0}^r (-1)^{|I|} d_I P_A^I. \quad (3.7)$$

Now one takes $\mathcal{L} \in \mathcal{F}(T_n^r)$ and constructs the total differential operator $P_{\mathcal{L}}$ according to:

$$P_{\mathcal{L}}(\xi) \equiv L_{pr(\xi)} \mathcal{L}. \quad (3.8)$$

Lemma 3.1 can be applied and immediately gives the following local formula:

$$P_{\mathcal{L}}(\xi) = \sum_{|I|=0}^r d_I \left(\xi^A \mathcal{E}_A^I(\mathcal{L}) \right) \quad (3.9)$$

where

$$\mathcal{E}_A^I(L) \equiv \sum_{|J| \leq r-|I|} (-1)^{|J|} \binom{|I|+|J|}{|I|} d_J \partial_A^{IJ} \mathcal{L} \quad (3.10)$$

are the so-called *Lie-Euler operators*; the Euler operator associated to $P_{\mathcal{L}}$ has the following expression:

$$E(P_{\mathcal{L}}) = \xi^A \mathcal{E}_A(L) \quad (3.11)$$

where

$$\mathcal{E}_A(L) \equiv \sum_{|J| \leq r} (-1)^{|J|} d_J \partial_A^J \mathcal{L} \quad (3.12)$$

are the *Euler-Lagrange expressions* associated to \mathcal{L} .

The proposition above leads to

Proposition 3.3 *If $\mathcal{L} \in \mathcal{F}(T_n^r)$, then there exists a globally defined 1-form, denoted by $\mathcal{E}(\mathcal{L})$ such that we have in the chart V^s , $s \geq 2r$:*

$$\mathcal{E}(\mathcal{L}) = \mathcal{E}_A(\mathcal{L}) dx^A. \quad (3.13)$$

Proof: By construction

$$Q_A = \mathcal{E}_A(\mathcal{L})$$

Now, one has from (3.4) :

$$\mathcal{E}_A(\mathcal{L}) = (\partial_A \bar{x}^B) \bar{\mathcal{E}}_B(\bar{\mathcal{L}}). \quad (3.14)$$

Combining with the transformation property

$$d\bar{x}^A = (\partial_B \bar{x}^A) dx^B$$

and obtains that the formula (3.13) has a global meaning. ■

One calls this form the *Euler-Lagrange form* associated to \mathcal{L} .

All the properties of this form listed in [12] are true in this case also. We insist only on the so-called *product rule* for the Euler-Lagrange expressions [1] which will be repeatedly used in the following.

Proposition 3.4 *If f and g are smooth functions on V^r then one has in V^s , $s \geq 2r$ the following formula:*

$$\mathcal{E}_A^I(fg) = \sum_{|J| \leq r-|I|} \binom{|I|+|J|}{|J|} \left[(d_J f) \mathcal{E}_A^{IJ}(g) + (d_J g) \mathcal{E}_A^{IJ}(f) \right], \quad |I| \leq r. \quad (3.15)$$

The proof goes by direct computation, directly from the definition of the Lie-Euler operators combined with Leibnitz rule of differentiation of a product.

We now come to the main definition. A smooth real function \mathcal{L} on $\text{Imm}T_n^r$ is called a *homogeneous Lagrangian* if it verifies the relation:

$$\mathcal{L}(x \cdot a) = \det(a) \mathcal{L}(x), \quad \forall a \in L_n^r; \quad (3.16)$$

here by $\det(a)$ we mean $\det(a_j^i)$.

Such an object induces on the factor manifold $P_n^r X$ an non-homogeneous object.

Proposition 3.5 *Let \mathcal{L} be a homogeneous Lagrangian. Then for every chart (W^r, Φ^r) on $\text{Imm}T_n^r X$ there exists a smooth real function on $\rho_n^r(W^r)$ such that:*

$$\mathcal{L} = \det(\mathbf{x}) L \circ \rho_n^r \quad (3.17)$$

and conversely, if \mathcal{L} is locally defined by this relation, then it verifies (3.16).

Proof: One chooses in (3.16) $a = \mathbf{x}$ (see (2.26)). ■

The function L is called the *non-homogeneous (local) Lagrangian* associated to \mathcal{L} .

As a consequence of the connection (3.17) we have

Proposition 3.6 *Let \mathcal{L} a homogeneous Lagrangian and (V, ψ) , $(\bar{V}, \bar{\psi})$ two overlapping charts on X . We consider on the associated charts (W^r, Φ^r) and $(\bar{W}^r, \bar{\Phi}^r)$ the corresponding non-homogeneous Lagrangians L and respectively \bar{L} . Then we have on the overlap $\rho_n^r(W^r \cap \bar{W}^r)$ the following formula:*

$$L = \mathcal{J} \bar{L} \quad (3.18)$$

where

$$\mathcal{J} \equiv \det(Q) = \det(D_i \bar{x}^j). \quad (3.19)$$

Proof: One writes (3.17) for both charts and gets

$$\mathcal{L} = \det(\mathbf{x}) L \circ \rho_n^r = \det(\bar{\mathbf{x}}) \bar{L} \circ \rho_n^r;$$

as a consequence

$$L \circ \rho_n^r = \det(\bar{\mathbf{x}} \mathbf{z}) \bar{L} \circ \rho_n^r.$$

One now uses the relation (2.50) and obtains the relation from the statement. ■

As consequence we have

Theorem 3.7 *One can globally define the equivalence class*

$$[\lambda] \in \Omega_n^r(PX) / \Omega_{n(c)}^r(PX)$$

such that the local expression of λ is

$$\lambda = L \theta_0; \quad (3.20)$$

here, as usual

$$\theta_0 \equiv dx^1 \wedge \dots \wedge dx^n. \quad (3.21)$$

Proof: One proves immediately that on the overlap of the associated charts from the proposition above one has:

$$\bar{\theta}_0 = \mathcal{J} \theta_0 + \text{contact terms}. \quad (3.22)$$

This result must be combined with (3.18) to obtain on $\rho_n^r(W^r \cap \bar{W}^r)$:

$$\lambda - \bar{\lambda} \in \Omega_{n(c)}^r(PX);$$

this proves the theorem. ■

It is natural to ask what is the connection between the Euler-Lagrange expression of the homogeneous and the corresponding non-homogeneous Lagrangian. The answer is contained in:

Theorem 3.8 *Suppose \mathcal{L} is a homogeneous Lagrangian defined on $\text{Imm}T_n^r X$ and L is the associated non-homogeneous Lagrangian. Then the following relations are valid on the chart W^s , $s \geq 2r$:*

$$\mathcal{E}_\sigma(\mathcal{L}) = \det(\mathbf{x}) E_\sigma(L) \circ \rho_n^s, \quad (3.23)$$

$$\mathcal{E}_\sigma^{j_1, \dots, j_k}(\mathcal{L}) = (-1)^k \det(\mathbf{x}) \sum_{|I| \geq k} (-1)^{|I|} \sum_{(I_1, \dots, I_k)} \mathcal{S}_{j_1, \dots, j_k}^+ z_{I_1}^{j_1} \dots z_{I_k}^{j_k} E_\sigma^I(T) \circ \rho_n^s, \quad (3.24)$$

and

$$\mathcal{E}_p(\mathcal{L}) = -\det(\mathbf{x}) y_p^\sigma E_\sigma(T) \circ \rho_n^s, \quad (3.25)$$

$$\mathcal{E}_p^j(T) = \det(\mathbf{x}) \left[z_p^j L + y_p^\sigma \sum_{I \leq r} z_I^j E_\sigma^I(L) \right] \circ \rho_n^s, \quad (3.26)$$

$$\mathcal{E}_p^{j_1, \dots, j_k}(\mathcal{L}) = (-1)^{k+1} \det(\mathbf{x}) y_p^\sigma \sum_{|I| \geq k} (-1)^{|I|} \sum_{(I_1, \dots, I_k)} \mathcal{S}_{j_1, \dots, j_k}^+ z_{I_1}^{j_1} \dots z_{I_k}^{j_k} E_\sigma^I(L) \circ \rho_n^s. \quad (3.27)$$

Proof:

(i) As a general strategy of the proof, we will try to transform the expression of the total differential operator $P_{\mathcal{L}}(\xi)$ in terms of L ; we have by definition

$$P_{\mathcal{L}}(\xi) = \sum_{|I| \leq r} (d_I \xi^A) \partial_A^I [\det(\mathbf{x}) L \circ \rho].$$

Using the chain rule one obtains rather easily from here:

$$\begin{aligned} P_{\mathcal{L}}(\xi) &= \det(\mathbf{x}) \sum_{k=0}^r (d_{j_1} \dots d_{j_k} \xi^\sigma) \sum_{l=k}^r (\Delta_\nu^{i_1, \dots, i_l} L) \circ \rho \quad (\partial_\sigma^{j_1, \dots, j_k} y_{i_1, \dots, i_l}^\nu) + \\ &\quad \det(\mathbf{x}) \sum_{k=0}^r (d_{j_1} \dots d_{j_k} \xi^p) \sum_{l=k}^r (\Delta_\nu^{i_1, \dots, i_l} L) \circ \rho \quad (\partial_p^{j_1, \dots, j_k} y_{i_1, \dots, i_l}^\nu) + \\ &\quad \det(\mathbf{x}) \xi^p (\partial_p L) \circ \rho + (d_j \xi^p) [\partial_p^j \det(\mathbf{x})] L \circ \rho \end{aligned} \quad (3.28)$$

We will now consider that the functions ξ^A depend only of the variables (x^i, y^σ) i.e. there exist the smooth real functions Ξ^A on W^r such that

$$\xi^A = \Xi^A \circ \rho. \quad (3.29)$$

(ii) To compute further the expression above one starts from (2.34) and firstly proves directly:

$$\partial_{\sigma}^{j_1, \dots, j_k} y_I^{\nu} = \delta_{\sigma}^{\nu} \mathcal{S}_{j_1, \dots, j_k}^{+} \sum_{(I_1, \dots, I_k)} z_{I_1}^{j_1} \cdots z_{I_k}^{j_k}, \quad 1 \leq k \leq |I| \leq r. \quad (3.30)$$

Next one proves

Lemma 3.9 *If we have $1 \leq k \leq |I| \leq r$ then one has:*

$$\partial_p^{j_1, \dots, j_k} y_I^{\sigma} = -\mathcal{S}_{j_1, \dots, j_k}^{+} \sum_{(I_0, \dots, I_k)} z_{I_1}^{j_1} \cdots z_{I_k}^{j_k} y_{pI_0}^{\sigma}. \quad (3.31)$$

Here we understand that the subsets I_1, \dots, I_k cannot be the emptyset; on the contrary, it is allowed to have $I_0 = \emptyset$.

Proof: By induction on $|I|$ starting from $|I| = k$. For this smallest possible value one uses (2.34). Then one supposes that the formula from the statement is valid for $k \leq |I| < r$, uses the defining recurrence relation (2.30) for the invariants and establishes the relation for iI . The cases $k = 1$ and $k > 1$ must be treated separately. ∇

Another auxiliary result is contained in the well known result:

Lemma 3.10 *The following formulæ are true:*

$$\partial_j^i \det(\mathbf{x}) = z_j^i \det(\mathbf{x}) \quad (3.32)$$

and

$$d_i[z_j^i \det(\mathbf{x})] = 0. \quad (3.33)$$

Proof: The first result is quite general i.e. valid for any invertible matrix and it is proved directly from the definition of the determinant. The second formula is a corollary of the first. ∇

One must use these results together with (2.46); we have from (3.28) after permuting the two summation signs:

$$\begin{aligned} P_{\mathcal{L}}(\xi) = & \det(\mathbf{x})[\xi^{\sigma}(\Delta_{\sigma}L) \circ \rho + \sum_{|I|=1}^r (\Delta_{\sigma}^I L) \circ \rho \sum_{k=1}^I \sum_{(I_1, \dots, I_k)} z_{I_1}^{j_1} \cdots z_{I_k}^{j_k} (d_{j_1} \cdots d_{j_k} \xi^{\sigma}) - \\ & \sum_{|I_0| \leq r} \sum_{|I|=1}^{r-|I_0|} \binom{|I| + |I_0|}{|I|} y_{pI_0}^{\sigma} (\Delta_{\sigma}^{I I_0} L) \circ \rho \sum_{k=1}^I \sum_{(I_1, \dots, I_k)} z_{I_1}^{j_1} \cdots z_{I_k}^{j_k} (d_{j_1} \cdots d_{j_k} \xi^p) + \\ & \xi^p(\partial_p L) \circ \rho - \xi^p(D_j L) \circ \rho] + d_j[\xi^p z_p^j \det(\mathbf{x}) L \circ \rho]. \quad (3.34) \end{aligned}$$

(iii) One can proceed further one must generalise the formula (2.46). We have the following formula

$$d_I(f \circ \rho) = \sum_{k=1}^I \sum_{(I_1, \dots, I_k)} x_{I_1}^{j_1} \cdots x_{I_k}^{j_k} (D_{j_1} \cdots D_{j_k} f) \circ \rho \quad (3.35)$$

which can be proved by induction on $|I|$.

This formula can be “inverted” rather easily and we get:

$$(D_I f) \circ \rho = \sum_{k=1}^I \sum_{(I_1, \dots, I_k)} z_{I_1}^{j_1} \cdots z_{I_k}^{j_k} d_{j_1} \cdots d_{j_k} (f \circ \rho). \quad (3.36)$$

The expression (3.34) can be considerably simplified to

$$P_{\mathcal{L}}(\xi) = \det(\mathbf{x}) \left[\sum_{|I| \leq 1} (D_I \Xi^\sigma) P_\sigma^I - (D_I \Xi^p) P_p^I \right] \circ \rho + d_j [\xi^p z_p^j \det(\mathbf{x}) L \circ \rho] \quad (3.37)$$

where we have defined

$$P_\sigma^I \equiv \Delta_\sigma^I L; \quad P_p^I \equiv \sum_{|J| \leq r-|I|} \binom{|I|+|J|}{|J|} y_{pJ}^\sigma \Delta_\sigma^{IJ}.$$

(iv) Now it is the time to apply lemma 3.1 and to obtain in this way:

$$P_{\mathcal{L}}(\xi) = \det(\mathbf{x}) \sum_{|I| \leq r} [D_I (\Xi^\sigma Q_\sigma^I - \Xi^p Q_p^I)] \circ \rho + d_j [\xi^p z_p^j \det(\mathbf{x}) L \circ \rho] \quad (3.38)$$

where one gets by some computations the following formulæ for the expressions Q_A^I :

$$Q_\sigma^I = \sum_{|J| \leq r-|I|} (-1)^{|J|} \binom{|I|+|J|}{|J|} D_J P_\sigma^{IJ} = E_\sigma^I(L) \quad (3.39)$$

and

$$Q_p^I = \sum_{|J| \leq r-|I|} (-1)^{|J|} \binom{|I|+|J|}{|J|} D_J P_p^{IJ} = y_p^\sigma E_\sigma^I(L). \quad (3.40)$$

(v) We want to compare the expression (3.38) with the right hand side of the formula from lemma 3.1. To do this we need one more combinatorial result valid for any smooth real function on the chart W^r :

$$\det(\mathbf{x}) (D_I f) \circ \rho = \sum_{k=1}^I (-1)^{r-|I|} d_{j_1} \cdots d_{j_k} \left[\det(\mathbf{x}) \sum_{(I_1, \dots, I_k)} z_{I_1}^{j_1} \cdots z_{I_k}^{j_k} (f \circ \rho) \right], \quad |I| \leq r. \quad (3.41)$$

One proves this formula by induction on $|I|$ and so the final expression for the total differential operator is

$$P_{\mathcal{L}}(\xi) = \det(\mathbf{x})(\Xi^\sigma Q_\sigma - \Xi^p Q_p^I) \circ \rho + d_j[\xi^p z_p^j \det(\mathbf{x}) L \circ \rho] + \sum_{k=1}^r (-1)^k d_{j_1} \cdots d_{j_k} \left[\det(\mathbf{x}) \sum_{|I| \geq k} (-1)^{|I|} \sum_{(I_1, \dots, I_k)} z_{I_1}^{j_1} \cdots z_{I_k}^{j_k} (\Xi^\sigma Q_\sigma^I - \Xi^p Q_p^I) \circ \rho \right] \quad (3.42)$$

If one uses the unicity statement from lemma 3.1 one obtains the desired formulæ. ■

Immediate consequences of the preceding theorem are

Corollary 3.11 *If \mathcal{L} is a homogeneous Lagrangian, then the following relations are true on the manifold $T_n^s X$, $s \geq 2r$:*

$$\mathcal{E}_A(\mathcal{L})(x \cdot a) = \det(a) \mathcal{E}_A(\mathcal{L})(x), \quad \forall a \in L_n^r. \quad (3.43)$$

In the conditions of the above theorem we have:

$$\mathcal{E}_A(\mathcal{L}) \equiv 0 \iff E_\sigma(L) \equiv 0. \quad (3.44)$$

We also have the analogue of proposition 3.6:

Proposition 3.12 *In the condition of the preceding theorem let us consider two overlapping charts (V, ψ) , $(\bar{V}, \bar{\psi})$. Then one has on the overlap of the associated charts: $\rho_n^s(W^s \cap \bar{W}^s)$ the following relation:*

$$E_\sigma(L) = \mathcal{J} P_\sigma^\nu \bar{E}_\nu(\bar{L}) \quad (3.45)$$

where the matrix P_σ^ν has been defined by the formula (2.66).

Proof:

We start from the transformation formula (3.14) for $A \rightarrow \sigma$:

$$\mathcal{E}_\sigma(\mathcal{L}) = (\Delta_\sigma \bar{x}^i) \bar{\mathcal{E}}_i(\bar{\mathcal{L}}) + (\Delta_\sigma \bar{y}^\nu) \bar{\mathcal{E}}_\nu(\bar{\mathcal{L}})$$

and substitute (3.23) and (3.25). Using (2.50) we obtain by elementary computations the relation from the statement. ■

Remark 3.12.1 *For a different proof of this result see [24].*

Now we have the analogue of theorem 3.7:

Theorem 3.13 *If \mathcal{L} is a homogeneous Lagrangian on T_n^r , then one can globally define the equivalence class*

$$[E(L)] \in \Omega_{n+1}^s(PX)/\Omega_{n+1(c)}^s(PX)$$

on P_n^s , $s \geq 2r$ such that the local expression of $E(L)$ is

$$E(L) = E_\sigma(L)\omega^\sigma \wedge \theta_0. \quad (3.46)$$

Proof: Follows the lines of theorem 3.7 and it is elementary. ■

We also note the following property:

Proposition 3.14 *If \mathcal{L} is a homogeneous Lagrangian on $T_n^r X$, then the corresponding Euler-Lagrange form verifies on $T_n^s X$, $s \geq 2r$ the following identity:*

$$(j_0^s \gamma)^* \mathcal{E}(\mathcal{L}) = 0, \quad \forall \gamma \in \text{Imm} T_n^s. \quad (3.47)$$

Proof: By direct computation we get

$$(j_0^s \gamma)^* \mathcal{E}(\mathcal{L}) = (\mathcal{E}_A(\mathcal{L})x_j^A) \circ j_0^s \gamma \quad dt^j.$$

Now one uses (3.23) and (3.25) to prove that the expression in the bracket is identically zero. ■

We close this subsection with some remarks.

Remark 3.14.1 *If \mathcal{L} is a homogeneous Lagrangian, one can expect some homogeneity property for the total differential operator associated to it. Indeed, one has for an arbitrary Lagrangian*

$$P_{\mathcal{L}} \circ \phi_a(\xi) = P_{\mathcal{L}}(\xi) \circ \phi_a \quad (3.48)$$

where ϕ_a denotes the right action of the differential group L_n^r .

As a consequence, one has for a homogeneous Lagrangian

$$P_{\mathcal{L}}(\xi) \circ \phi_a = \det(a) P_{\mathcal{L}}(\xi), \quad \forall a \in L_n^r. \quad (3.49)$$

3.2 Differential Equations on Grassmann Manifolds

An element $\mathcal{T} \in \Omega_{n+1}^s(X)$ is called a *differential equation* on $T_n^s X$. In the chart (V^s, ψ^s) the differential equation \mathcal{T} has the following local expression:

$$\mathcal{T} = \mathcal{T}_A dx^A. \quad (3.50)$$

It is clear that the Euler-Lagrange form defined by (3.13) is a differential equation. A differential equation \mathcal{T} is called *variational* if there exists a Lagrangian \mathcal{L} on $T_n^s X$, $s \geq 2r$ such that we have $\mathcal{T} = \mathcal{E}(\mathcal{L})$. If the function \mathcal{L} is only locally defined, then such a differential equation is called *locally variational* (or, of the *Euler-Lagrange type*).

If $\gamma : \mathbb{R}^n \rightarrow X$ is a immersion, then one says that the differential equation \mathcal{T} *verifies the differential equation iff* we have

$$(j_0^s \gamma)^* i_Z \mathcal{T} = 0 \quad (3.51)$$

for any vector field Z on $J_n^s X$. In local coordinates we have on V^s :

$$\mathcal{T}_A \circ j_0^s \gamma = 0 \quad (A = 1, \dots, N). \quad (3.52)$$

Guided by corollary 3.11 and prop 3.14 we also introduce the following definition. We say that \mathcal{T} is a *homogeneous differential equation* on $\text{Imm} T_n^s X$ if it verifies the following conditions:

$$(\phi_a)^* \mathcal{T} = \det(a) \mathcal{T}, \quad \forall a \in L_n^s \quad (3.53)$$

and

$$(j_0^s \gamma)^* \mathcal{T} = 0, \quad \forall \gamma \in \text{Imm} T_n^s. \quad (3.54)$$

Then we have the following result which can be proved by elementary computations suggested by the similar results obtained for a differential equation of the Euler-Lagrange type.

Theorem 3.15 *Let \mathcal{T} be a homogeneous differential equation on $\text{Imm} T_n^s X$. Then there exist some local smooth real functions T_σ in every chart $\rho_n^s(W^s)$ such that one has:*

$$\mathcal{T}_\sigma = \det(\mathbf{x}) T_\sigma \circ \rho, \quad \mathcal{T}_i = -\det(\mathbf{x}) y_i^\sigma T_\sigma \circ \rho. \quad (3.55)$$

As a consequence, if (V, ψ) , $(\bar{V}, \bar{\psi})$ are two overlapping charts on X , then one has on the intersection $\rho_n^s(W^s \cap \bar{W}^s)$ the following transformation formula:

$$T_\sigma = \mathcal{J} P_\sigma^\nu \bar{T}_\nu \quad (3.56)$$

and the class

$$[T] \in \Omega_{n+1}^s(PX) / \Omega_{n+1(c)}^s(PX)$$

can be properly defined such that we have locally

$$T = T_\sigma \omega^\sigma \wedge \theta_0. \quad (3.57)$$

One calls T the *associated (local) non-homogenous differential equation*.

We now prove the existence of the (globally) defined Helmholtz-Sonin form associated to a differential equation. By analogy with [12] we have the following result

Theorem 3.16 *Let \mathcal{T} be a differential equation on $T_n^s X$ with the local form given by (3.50). We define the following expressions in any chart V^t , $t > 2s$:*

$$\mathcal{H}_{AB}^J \equiv \partial_B^J \mathcal{T}_A - (-1)^{|J|} \mathcal{E}_A^J(\mathcal{T}_B), \quad |J| \leq s. \quad (3.58)$$

Then there exists a globally defined 2-form, denoted by $\mathcal{H}(\mathcal{T})$ such that in any chart V^t we have:

$$\mathcal{H}(\mathcal{T}) = \sum_{|J| \leq s} \mathcal{H}_{AB}^J dx_J^B \wedge dx^A. \quad (3.59)$$

Proof: We sketch briefly the argument from [12]. Let ξ be a vector field on X ; we define a (global) 1-form $\mathcal{H}_\xi(\mathcal{T})$ according to:

$$\mathcal{H}_\xi(\mathcal{T}) \equiv L_{pr(\xi)} \mathcal{T} - \mathcal{E}(i_{pr(\xi)} \mathcal{T}) \quad (3.60)$$

and the following local expression is obtained:

$$\mathcal{H}_\xi(\mathcal{T}) = \sum_{|I| \leq s} (d_I \xi^B) \mathcal{H}_{AB}^I dx^A. \quad (3.61)$$

The transformation formula at a change of charts for the expressions $d_I \xi^B$ is:

$$\bar{d}_I \bar{\xi}^A = \sum_{|J| \leq |I|} \left(\partial_B^J \bar{x}_I^A \right) (d_J \xi^B), \quad |I| = 0, \dots, s. \quad (3.62)$$

Using the transformation formula (3.62) one can obtain the transformation formula for the expressions \mathcal{H}_{AB}^I : one has in the overlap $V^t \cap \bar{V}^t$, $t \geq 2s$:

$$\mathcal{H}_{DB}^J = \sum_{|I| \geq |J|} \left(\partial_B^J \bar{x}_I^A \right) \left(\partial_D \bar{x}^C \right) \bar{\mathcal{H}}_{CA}^I. \quad (3.63)$$

This transformation formula leads now to the fact that $\mathcal{H}(\mathcal{T})$ has an invariant meaning. ■

$\mathcal{H}(\mathcal{T})$ is called the *Helmholtz-Sonin form* associated to \mathcal{T} and \mathcal{H}_{AB}^I are the *Helmholtz-Sonin expressions* associated to \mathcal{T} .

A well-known corollary of the theorem above is:

Corollary 3.17 *The differential equation \mathcal{T} is locally variational iff $\mathcal{H}(\mathcal{T}) = 0$ iff*

$$\mathcal{H}_{AB}^I = 0, \quad \forall A, B = 1, \dots, N, \quad \forall |I| \leq r. \quad (3.64)$$

The proof is identical with the one presented in [12]. The preceding equations are called the *Helmholtz-Sonin equations*.

As in the preceding subsection, if \mathcal{T} is a homogeneous differential equation, we have a very precise connection between the Helmholtz-Sonin expressions of \mathcal{T} and of T from theorem 3.15.

Theorem 3.18 *Suppose \mathcal{T} is a homogeneous equation defined on $\text{Imm}T_n^s X$ and T_σ the components of the associated non-homogeneous equation. Then the following relations are valid on the chart W^t , $t \geq 2s$:*

$$\mathcal{H}_{\sigma\nu}(\mathcal{T}) = \det(\mathbf{x}) H_{\sigma\nu}(T) \circ \rho_n^t, \quad (3.65)$$

$$\mathcal{H}_{\sigma\nu}^{j_1, \dots, j_k}(\mathcal{T}) = \det(\mathbf{x}) \sum_{|I| \geq k} \sum_{(I_1, \dots, I_k)} \mathcal{S}_{j_1, \dots, j_k}^+ z_{I_1}^{j_1} \dots z_{I_k}^{j_k} H_{\sigma\nu}^I(T) \circ \rho_n^t, \quad k = 1, \dots, s. \quad (3.66)$$

$$\mathcal{H}_{\sigma p}(\mathcal{T}) = -\det(\mathbf{x}) \sum_{I_0 \leq s} y_{pI_0}^\nu H_{\sigma\nu}^{I_0}(T) \circ \rho_n^t, \quad (3.67)$$

$$\mathcal{H}_{\sigma p}^{j_1, \dots, j_k}(\mathcal{T}) = -\det(\mathbf{x}) \sum_{I_0 \leq s} \sum_{|I|=k}^{s-|I_0|} \binom{|I| + |I_0|}{|I|} \sum_{(I_1, \dots, I_k)} \mathcal{S}_{j_1, \dots, j_k}^+ z_{I_1}^{j_1} \dots z_{I_k}^{j_k} y_{pI_0}^\nu H_{\sigma\nu}^{II_0}(T) \circ \rho_n^t, \quad k = 1, \dots, s, \quad (3.68)$$

$$\mathcal{H}_{p\nu}(\mathcal{T}) = -\det(\mathbf{x}) y_p^\sigma H_{\sigma\nu}(T) \circ \rho_n^t, \quad (3.69)$$

$$\mathcal{H}_{p\nu}^{j_1, \dots, j_k}(\mathcal{T}) = -\det(\mathbf{x}) y_p^\sigma \sum_{I_0 \leq s} \sum_{|I| \geq k} \sum_{(I_1, \dots, I_k)} \mathcal{S}_{j_1, \dots, j_k}^+ z_{I_1}^{j_1} \dots z_{I_k}^{j_k} H_{\sigma\nu}^I(T) \circ \rho_n^t, \quad k = 1, \dots, s \quad (3.70)$$

and

$$\mathcal{H}_{pq}(\mathcal{T}) = \det(\mathbf{x}) y_p^\sigma \sum_{I_0 \leq s} y_{qI_0}^\nu H_{\sigma\nu}^{I_0}(T) \circ \rho_n^t, \quad (3.71)$$

$$\mathcal{H}_{pq}^{j_1, \dots, j_k}(\mathcal{T}) = \det(\mathbf{x}) y_p^\sigma \sum_{I_0 \leq s} \sum_{|I|=k}^{s-|I_0|} \binom{|I| + |I_0|}{|I|} \sum_{(I_1, \dots, I_k)} \mathcal{S}_{j_1, \dots, j_k}^+ z_{I_1}^{j_1} \dots z_{I_k}^{j_k} y_{qI_0}^\nu H_{\sigma\nu}^{II_0}(T) \circ \rho_n^t, \quad k = 1, \dots, s. \quad (3.72)$$

The proof is tedious but elementary. One must use the formulæ derived in theorem 3.8 combined with the derivation property from proposition 3.4 to prove case by case the formulæ from the statement. Occasionally, one must study separately the cases $k = 0, 1$ and $k > 1$.

In the conditions of the above theorem we have:

Corollary 3.19

$$\mathcal{H}_{AB}^I(T) \equiv 0 \iff H_{\sigma\nu}(T) \equiv 0. \quad (3.73)$$

As it can be expected we have the analogues of propositions 3.6 and 3.12:

Proposition 3.20 *In the condition of the preceding theorem let us consider two overlapping charts (V, ψ) , $(\bar{V}, \bar{\psi})$. Then one has on the overlap of the associated charts: $\rho_n^t(W^t \cap \bar{W}^t)$ the following relations:*

$$H_{\sigma\nu}(T) = \mathcal{J}P_\sigma^\alpha \sum_{|J| \leq s} \left[(\Delta_\nu \bar{y}_J^\beta) - P_j^i (D_i \bar{y}_J^\beta) (\Delta_\nu \bar{x}^j) \right] \bar{H}_{\alpha\beta}^J(\bar{T}) \quad (3.74)$$

$$H_{\sigma\nu}^I(T) = \mathcal{J}P_\sigma^\alpha \sum_{|J| \geq |I|} (\Delta_\nu \bar{y}_J^\beta) \bar{H}_{\alpha\beta}^J(\bar{T}), \quad \forall I \neq \emptyset. \quad (3.75)$$

Proof: (i) It is convenient to introduce the expression g_{j_1, \dots, j_k} , $k = 0, \dots, r$ completely symmetric in all indices (with the convention $g_\emptyset = g$) and to use the (3.65) and (3.66) to obtain:

$$\begin{aligned} \det(\mathbf{x}) \sum_{|I| \leq s} (g \cdot \mathbf{z})_I H_{\sigma\nu}^I(T) \rho &= \\ \sum_{k=0}^s g_{j_1, \dots, j_k} \mathcal{H}_{\sigma\nu}^{j_1, \dots, j_k}(T) &= \\ \sum_{|I| \leq s} g_I \sum_{|J| \geq |I|} (\partial_\sigma^J \bar{x}_I^A) (\partial_\nu \bar{x}^C) \bar{\mathcal{H}}_{CA}^I(\bar{T}) \end{aligned}$$

where use have been made of the transformation formula (3.63) for $B \rightarrow \sigma$, $D \rightarrow \nu$.

If we make here $g \rightarrow g \cdot \mathbf{x}$ we obtain equivalently after elementary prelucrations of the formula above:

$$\begin{aligned} \det(\mathbf{x}) H_{\sigma\nu}(T) \circ \rho &= \sum_{|I| \leq s} \sum_{|J| \geq |I|} [(\partial_\nu \bar{x}_I^\beta) (\partial_\sigma \bar{x}^\alpha) \bar{\mathcal{H}}_{\alpha\beta}^I(\bar{T}) + (\partial_\nu \bar{x}_I^\beta) (\partial_\sigma \bar{x}^p) \bar{\mathcal{H}}_{p\beta}^I(\bar{T}) + \\ &(\partial_\nu \bar{x}_I^q) (\partial_\sigma \bar{x}^\alpha) \bar{\mathcal{H}}_{\alpha q}^I(\bar{T}) + (\partial_\nu \bar{x}_I^q) (\partial_\sigma \bar{x}^p) \bar{\mathcal{H}}_{pq}^I(\bar{T})] \end{aligned} \quad (3.76)$$

and for $k \geq 1$:

$$\begin{aligned} \det(\mathbf{x}) H_{\sigma\nu}^{j_1, \dots, j_k}(T) \circ \rho &= \sum_{|J| \geq k} \sum_{(J_1, \dots, J_k)} \mathcal{S}_{j_1, \dots, j_k}^+ x_{J_1}^{j_1} \dots x_{J_k}^{j_k} \sum_{|J| \geq |I|} [(\partial_\nu \bar{x}_I^\beta) (\partial_\sigma \bar{x}^\alpha) \bar{\mathcal{H}}_{\alpha\beta}^I(\bar{T}) + \\ &(\partial_\nu \bar{x}_I^\beta) (\partial_\sigma \bar{x}^p) \bar{\mathcal{H}}_{p\beta}^I(\bar{T}) + (\partial_\nu \bar{x}_I^q) (\partial_\sigma \bar{x}^\alpha) \bar{\mathcal{H}}_{\alpha q}^I(\bar{T}) + (\partial_\nu \bar{x}_I^q) (\partial_\sigma \bar{x}^p) \bar{\mathcal{H}}_{pq}^I(\bar{T})] \end{aligned} \quad (3.77)$$

The second relation can be considerably simplified if one uses (2.31), more precisely the consequence

$$\Delta_{\nu}^{j_1, \dots, j_k} x_I^{\mu} = \delta_{\nu}^{\mu} \sum_{(I_1, \dots, I_k)} \mathcal{S}_{j_1, \dots, j_k}^{+} x_{I_1}^{j_1} \dots x_{I_k}^{j_k}; \quad (3.78)$$

the chain rule gives from here

$$\Delta_{\nu}^{j_1, \dots, j_k} \bar{x}_I^A = \sum_{|J|=k}^{[I]} (\partial_{\nu}^J \bar{x}_I^A) \sum_{(J_1, \dots, J_k)} \mathcal{S}_{j_1, \dots, j_k}^{+} x_{J_1}^{j_1} \dots x_{J_k}^{j_k}. \quad (3.79)$$

So, the relation (3.77) becomes:

$$\begin{aligned} \det(\mathbf{x}) H_{\sigma\nu}^{j_1, \dots, j_k}(T) \circ \rho = \sum_{|I| \geq k} \{ & (\Delta_{\nu}^{j_1, \dots, j_k} \bar{x}_I^{\beta}) [(\partial_{\sigma} \bar{x}^{\alpha}) \bar{\mathcal{H}}_{\alpha\beta}^I(\bar{T}) + \\ & (\partial_{\sigma} \bar{x}_I^p) \bar{\mathcal{H}}_{p\beta}^I(\bar{T})] + (\Delta_{\nu}^{j_1, \dots, j_k} \bar{x}_I^q) [(\partial_{\sigma} \bar{x}^{\alpha}) \bar{\mathcal{H}}_{\alpha q}^I(\bar{T}) + (\partial_{\sigma} \bar{x}^p) \bar{\mathcal{H}}_{pq}^I(\bar{T})] \} \end{aligned} \quad (3.80)$$

If we compare with (3.76) we see that the preceding relation stays true for $k = 0$ also.

Now we use again the theorem above in the right hand side of the relation just derived and obtains after some computations (using the relations (3.30) and (3.31) and the chain rule) the relations from the statement of the theorem. ■

Now we have the analogue of theorems 3.7 and 3.13

Theorem 3.21 *If \mathcal{T} is a homogeneous differential equation on $\text{Imm}T_n^s$, then one can globally define the equivalence class*

$$[H(T)] \in \Omega_{n+2}^t(PX) / \Omega_{n+2(c)}^t(PX)$$

on P_n^t , $t \geq 2s$ such that the local expression of $H(T)$ is

$$H(T) = \sum_{|I| \leq s} H_{\sigma\nu}^I(L) \omega_I^{\nu} \wedge \omega^{\sigma} \wedge \theta_0. \quad (3.81)$$

Proof: Follows the lines of theorem 3.13 and it is elementary. ■

As a consequence of the theorems 3.7, 3.13 and 3.21 we can apply the exactness of the variational sequence and obtain that the expressions of an arbitrary variationally trivial Lagrangian and of a locally variational differential equation are coinciding with the expressions derived in [13].

4 Lagrangian Formalism on Second Order Grassmann Bundles

4.1 The Second Order Grassmann Bundle

Here we particularize the results obtained in the preceding sections for the case $r = 2$.

The coordinates on $T_n^2 X$ are (x^A, x_j^A, x_{ij}^A) and with the help of the derivative operators (see (2.3))

$$\partial_A \equiv \frac{\partial}{\partial x^A}, \quad \partial_A^j \equiv \frac{\partial}{\partial x_j^A}, \quad \partial_A^{ij} \equiv \begin{cases} \frac{\partial}{\partial x_{ij}^A}, & \text{for } i = j \\ \frac{1}{2} \frac{\partial}{\partial x_{ij}^A}, & \text{for } i \neq j \end{cases} \quad (4.1)$$

we have for any smooth function f (see (2.6)):

$$df = (\partial^A f) dx^A + (\partial_i^A f) dx_i^A + (\partial_{ij}^A f) dx_{ij}^A \quad (4.2)$$

We have the following formulæ (see (2.4)):

$$\partial_A x^B = \delta_A^B, \quad \partial_A^i x_j^B = \delta_A^B \delta_j^i, \quad \partial_A^{ij} x_{lm}^B = \frac{1}{2} \delta_A^B (\delta_l^i \delta_m^j + \delta_m^i \delta_l^j) \quad (4.3)$$

and the other derivatives are zero. The formal derivatives (see (2.7)) are in this case:

$$d_i^r \equiv x^A \partial_A + x_{ij}^A \partial_A^j \quad (4.4)$$

and from here we immediately have (see (2.8)):

$$d_i x^A = x_i^A, \quad d_i x_j^A = x_{ij}^A. \quad (4.5)$$

The formulæ for the induces change of charts (see (2.15)) are in this case:

$$F_i^A = x_i^B \partial_B F^A, \quad F_{i_1, i_2}^A = x_{i_1, i_2}^B \partial_B F^A + x_{i_1}^{B_1} x_{i_2}^{B_2} \partial_{B_1} \partial_{B_2} F^A \quad (4.6)$$

The elements of the differential group are of the form

$$a = (a_i^j, a_{i_1, i_2}^j), \quad \det(a_i^j) \neq 0 \quad (4.7)$$

with the composition law (see (2.21)):

$$(a \cdot b)_i^k = b_j^k a_i^j, \quad (a \cdot b)_{i_1, i_2}^k = b_{i_1, i_2}^j a_j^k + b_{i_1}^{j_1} b_{i_2}^{j_2} a_{j_1, j_2}^k \quad (4.8)$$

and the inverse element given by

$$a^{-1} = ((a^{-1})_i^j, -(a^{-1})_k^j (a^{-1})_{i_1}^{j_1} (a^{-1})_{i_2}^{j_2} a_{j_1, j_2}^k). \quad (4.9)$$

The action of this group on $T_n^2 X$ is (see (2.23)):

$$(a \cdot x)^A = x^A, \quad (a \cdot x)_i^A = a_i^j x_j^A, \quad (a \cdot x)_{i_1, i_2}^A = a_{i_1, i_2}^j x_j^A + a_{i_1}^{j_1} a_{i_2}^{j_2} x_{j_1, j_2}^A. \quad (4.10)$$

The expressions for the invariants of this action are (see (2.34)):

$$y^\sigma = x^\sigma, \quad y_i^\sigma = z_i^j x_j^\sigma, \quad y_{i_1, i_2}^\sigma = z_{i_1}^{j_1} z_{i_2}^{j_2} (x_{j_1, j_2}^\sigma - z_p^k y_k^\sigma x_{j_1, j_2}^p) \quad (4.11)$$

and they are, together with x^i , local coordinates on $P_n^2 X$.

The inverse of these formulæ are (see (2.31)):

$$x^\sigma = y^\sigma, \quad x_i^\sigma = x_i^j y_j^\sigma, \quad x_{i_1, i_2}^\sigma = x_{i_1}^{j_1} x_{i_2}^{j_2} y_{j_1, j_2}^\sigma + y_k^\sigma x_{i_1, i_2}^k \quad (4.12)$$

On the factor manifold $P_n^2 X$ one introduces the derivatives operators (see (2.43) and (2.44)):

$$\Delta_\sigma \equiv \frac{\partial}{\partial y^\sigma}, \quad \Delta_\sigma^j \equiv \frac{\partial}{\partial y_j^\sigma}, \quad \Delta_\sigma^{ij} \equiv \begin{cases} \frac{\partial}{\partial y_{ij}^\sigma}, & \text{for } i = j \\ \frac{1}{2} \frac{\partial}{\partial y_{ij}^\sigma}, & \text{for } i \neq j \end{cases} \quad (4.13)$$

and

$$D_i \equiv \frac{\partial}{\partial x^i} + y_i^\sigma \Delta_\sigma + y_{ij}^\sigma \Delta_\sigma^j. \quad (4.14)$$

The formula for the change of charts on $P_n^2 X$ is (see (2.48)):

$$\bar{y}_i^\sigma = P_i^j D_j \bar{y}^\sigma, \quad \bar{y}_{i_1, i_2}^\sigma = P_{i_1}^{j_1} P_{i_2}^{j_2} \left[-P_l^m (D_{j_1} D_{j_2} \bar{x}^l) (D_m \bar{y}^\sigma) + D_{j_1} D_{j_2} \bar{y}^\sigma \right]. \quad (4.15)$$

Finally, the expressions for the contact forms are (see (2.59)):

$$\omega^\sigma \equiv dy^\sigma - y_i^\sigma dx^i, \quad \omega_j^\sigma \equiv dy_j^\sigma - y_{ij}^\sigma dx^i \quad (4.16)$$

and their transformation for a change of charts is (see (2.64) and (2.65)):

$$\bar{\omega}^\sigma = P_\nu^\sigma \omega^\nu, \quad \bar{\omega}_i^\sigma = P_\nu^\sigma P_i^j \bar{\omega}_j^\nu + \left[\Delta_\nu \bar{y}_i^\sigma - P_k^l (\Delta_\nu \bar{x}^k) (D_l \bar{y}_i^\sigma) \right] \omega^\nu. \quad (4.17)$$

4.2 Lagrangian Formalism

Here we give a different approach to the Lagrangian formalism based on a certain $(n+1)$ -form defined on the Grassmann manifold P_n^2 [15]. The description of the formalism will be slightly different a some new material will appear.

As in [15], [11], we base our formalism on the operator K (see prop. 2.20) which in our case is defined on $\Omega_{q, hor}^2(PX)$ and is given by

$$K\alpha \equiv i_{D_j} i_{\Delta_\sigma^j} (\omega^\sigma \wedge \alpha). \quad (4.18)$$

We define the space of *Lagrange-Souriau forms* according to

$$\Omega_{LS}^2 \equiv \{\alpha \in \Omega_{n+1, hor}^2(PX) | d\alpha = 0, \quad K\alpha = 0, \quad i_{V_1} i_{V_2} \alpha = 0, \quad \forall V_i \in \text{Vert}(X)\}. \quad (4.19)$$

where $\text{Vert}(X)$ is the space of vertical vector fields on P_n^2 with respect to the projection $\rho_n^{2,1}$.

By definition, a *Lagrangian system* on P_n^2 is a couple (E, α) where E is a open sub-bundle of P_n^2 and α is a Lagrange-Souriau form.

If $\gamma \in \text{Imm} T_n^2$ we say that it *verifies the Euler-Lagrange equations* iff

$$[j_0^2 \gamma]^* \alpha = 0. \quad (4.20)$$

It is easy to see that the local expression of a Lagrange-Souriau form is

$$\begin{aligned} \alpha = & \sum_{k=0}^n \frac{1}{k!} F_{\sigma_0, \dots, \sigma_k}^{i_0, \dots, i_k} \omega_{i_0}^{\sigma_0} \wedge \omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_k} \wedge \theta_{i_1, \dots, i_k} + \\ & \sum_{k=0}^n \frac{1}{(k+1)!} E_{\sigma_0, \dots, \sigma_k}^{i_1, \dots, i_k} \omega^{\sigma_0} \wedge \omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_k} \wedge \theta_{i_1, \dots, i_k} \end{aligned} \quad (4.21)$$

where we have defined:

$$\theta_{i_1, \dots, i_k} \equiv \binom{n}{k} \varepsilon_{i_1, \dots, i_n} dx^{i_{k+1}} \wedge \dots \wedge dx^{i_n}, \quad k = 0, \dots, n. \quad (4.22)$$

We can admit, without loosing generality, some (anti)-symmetry properties.

$$F_{\sigma_0, \sigma_{Q(1)}, \dots, \sigma_{Q(k)}}^{i_0, i_{P(1)}, \dots, i_{P(k)}} = (-1)^{|P|+|Q|} F_{\sigma_0, \sigma_1, \dots, \sigma_k}^{i_0, i_1, \dots, i_k}, \quad \forall P, Q \in \mathcal{P}_k \quad (4.23)$$

and

$$E_{\sigma_{Q(0)}, \dots, \sigma_{Q(k)}}^{i_{P(1)}, \dots, i_{P(k)}} = (-1)^{|P|+|Q|} E_{\sigma_0, \sigma_1, \dots, \sigma_k}^{i_1, \dots, i_k}, \quad \forall P \in \mathcal{P}_k, \forall Q \in \mathcal{P}_{k+1}. \quad (4.24)$$

The condition

$$K\alpha = 0$$

appearing in the definition (4.19) of a Lagrange-Souriau form, has the following local form (see [11]):

$$\mathcal{S}_{i_0, \dots, i_k}^- \mathcal{S}_{\sigma_0, \dots, \sigma_k}^- F_{\sigma_0, \dots, \sigma_k}^{i_0, \dots, i_k} = 0, \quad k = 1, \dots, n \quad (4.25)$$

where $\mathcal{S}_{\sigma_0, \dots, \sigma_k}^\pm$ are defined similarly to (2.5).

The local form of the Euler-Lagrange equation (4.20) is simply:

$$E_\sigma \circ [j_0^2 \gamma] = 0 \quad (4.26)$$

Remark 4.0.1 For the case $n = 1$, the 2-form α above appears in [29] and the condition (4.25) is investigated in [17] and [16].

The justification of the terminology for (4.26) is contained in the following result:

Proposition 4.1 The expressions E_σ verify the Helmholtz equations (3.73).

Proof: One writes in detail the closedness condition $d\alpha = 0$ and find out, in particular, the following equations:

$$d_j E_{\sigma_0, \sigma_1}^j + \partial_{\sigma_0} E_{\sigma_1} - \partial_{\sigma_1} E_{\sigma_0} = 0, \quad (4.27)$$

$$d_l F_{\nu,\sigma}^{jl} - \partial_\sigma F_\nu^j + \partial_\nu^j E_\sigma + E_{\nu,\sigma}^j = 0, \quad (4.28)$$

$$\partial_\nu^{jk} E_\sigma + \frac{1}{2} (F_{\nu,\sigma}^{jk} + F_{\nu,\sigma}^{kj}) = 0, \quad (4.29)$$

$$F_{\sigma,\nu}^{jl} - F_{\nu,\sigma}^{lj} = \partial_\nu^l F_\sigma^j - \partial_\sigma^j F_\nu^l. \quad (4.30)$$

From (4.25) we get, in particular:

$$F_\sigma^j = 0 \quad (4.31)$$

and

$$F_{\nu,\sigma}^{jl} - F_{\sigma,\nu}^{jl} - F_{\nu,\sigma}^{lj} + F_{\sigma,\nu}^{lj} = 0. \quad (4.32)$$

We use (4.31) in (4.28) and get:

$$d_l F_{\nu,\sigma}^{jl} + \partial_\nu^j E_\sigma + E_{\nu,\sigma}^j = 0,$$

If we substitute the last term of this relation into (4.27) and use (4.29) we get:

$$\partial_{\sigma_0} E_{\sigma_1} - \partial_{\sigma_1} E_{\sigma_0} = d_j \partial_{\sigma_0}^j E_{\sigma_1} - d_j d_l \partial_{\sigma_0}^{jl} E_{\sigma_1}. \quad (4.33)$$

Next, we take the symmetric part in ν, σ of (4.28) and obtain:

$$\partial_\nu^j E_\sigma + \partial_\sigma^j E_\nu = -d_l (F_{\nu,\sigma}^{jl} + F_{\sigma,\nu}^{jl}).$$

One uses here (4.32) and next (4.30) + (4.31) to get:

$$\partial_\nu^j E_\sigma + \partial_\sigma^j E_\nu = 2d_l \partial_\sigma^{jl} E_\nu. \quad (4.34)$$

Finally, the antisymmetric part of (4.29) in σ, ν is:

$$\partial_\nu^{jk} E_\sigma = \partial_\sigma^{jk} E_\nu. \quad (4.35)$$

The equations (4.33), (4.34) and (4.35) are the Helmholtz-Sonin equations for the expressions E_σ . ■

We now mention a result derived in [15]:

Proposition 4.2 *There exists in every chart a local n -form β on $P_n^1 X$ having the local expression*

$$\beta = \sum_{k=0}^n \frac{1}{k!} L_{\sigma_1, \dots, \sigma_k}^{i_1, \dots, i_k} \omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_k} \wedge \theta_{i_1, \dots, i_k} \quad (4.36)$$

where

$$L_{\sigma_1, \dots, \sigma_k}^{i_1, \dots, i_k} = \mathcal{S}_{i_1, \dots, i_k}^- \mathcal{S}_{\sigma_1, \dots, \sigma_k}^- \partial_{\sigma_1}^{i_1} \dots \partial_{\sigma_k}^{i_k} L, \quad k = 0, \dots, n \quad (4.37)$$

such that:

$$\alpha = d(\rho_n^{2,1})^* \beta. \quad (4.38)$$

Remark 4.2.1 The form β is a generalization of the Poincaré-Cartan form [6], [25]. It had appeared in the literature in [19], [4], [5], [27]. For other generalisations of the Poincaré-Cartan form see [26], [18], [9], [8], [10], [28] and [20] where the notion of Lepage is introduced for such generalisations.

As a consequence we can express the coefficients of the form α given by (4.21) in terms of the smooth function L [11]:

$$F_{\sigma_0, \dots, \sigma_k}^{i_0, \dots, i_k} = \partial_{\sigma_0}^{i_0} L_{\sigma_1, \dots, \sigma_k}^{i_1, \dots, i_k} - L_{\sigma_0, \dots, \sigma_k}^{i_0, \dots, i_k}, \quad k = 0, \dots, n \quad (4.39)$$

and

$$E_{\sigma_0, \dots, \sigma_k}^{i_1, \dots, i_k} = \mathcal{S}_{i_0, \dots, i_k}^- \partial_{\sigma_0} L_{\sigma_1, \dots, \sigma_k}^{i_1, \dots, i_k} - d_{i_0} L_{\sigma_0, \dots, \sigma_k}^{i_0, \dots, i_k}, \quad k = 0, \dots, n. \quad (4.40)$$

In particular, for $k = 0$ the preceding formula is

$$E_\sigma = \partial_\sigma L - d_i \partial_\sigma^i L \quad (4.41)$$

i.e. the Euler-Lagrange operator. This is another justification of the terminology for the equations (4.20) (and (4.26).) This shows that the expressions $E_{\sigma_0, \dots, \sigma_k}^{i_1, \dots, i_k}$, $k = 0, \dots, n$ are also some generalizations of the Euler-Lagrange expressions, however, different from the Lie-Euler expressions introduced in [1] and given by (3.10).

The coefficients of α verify some recurrence relations:

$$E_{\sigma_0, \dots, \sigma_k}^{i_1, \dots, i_k} = -\mathcal{S}_{i_1, \dots, i_k}^- \mathcal{S}_{\sigma_0, \dots, \sigma_k}^- \left(\partial_{\sigma_0}^{i_1} + d_l \partial_{\sigma_0}^{l i_1} \right) E_{\sigma_1, \dots, \sigma_k}^{i_2, \dots, i_k}, \quad k = 1, \dots, n \quad (4.42)$$

and

$$F_{\sigma_0, \dots, \sigma_k}^{i_0, \dots, i_k} = \text{const} \times \mathcal{S}_{\sigma_0, \dots, \sigma_k}^- \mathcal{S}_{i_0, \dots, i_k}^- \partial_{\sigma_0}^{i_0} F_{\sigma_1, \dots, \sigma_k}^{i_1, \dots, i_k}, \quad k = 2, \dots, n. \quad (4.43)$$

Proof: By exploiting the conditions $d\alpha = 0$ and $K\alpha = 0$ written in local coordinates. Alternatively, one computes the right hand sides of these formulæ using (4.39) and (4.40) and obtains the left hand sides. ■

These recurrence relations have an important consequence:

Corollary 4.3 *In the conditions above we have*

$$\alpha \equiv 0 \iff E_\sigma \equiv 0. \quad (4.44)$$

We close this subsection giving the connection between the form α introduced here and the Lagrange-Souriau form σ introduced in [11], [15]. We have

Proposition 4.4 *Let α be a Lagrange-Souriau form. Then there exists a $(n+1)$ -form σ on P_n^1 such that*

$$\alpha = (\rho_n^{2,1})^* \sigma. \quad (4.45)$$

Proof: One explicitates the condition

$$d\alpha = 0$$

appearing in the definition of a Lagrange-Souriau form (4.19) and finds out in particular:

$$\partial_\nu^{jk} F_{\sigma_0, \dots, \sigma_k}^{i_0, \dots, i_k} = 0, \quad k = 0, \dots, n \quad (4.46)$$

$$\partial_\nu^{jk} E_{\sigma_0, \dots, \sigma_k}^{i_1, \dots, i_k} + \frac{1}{2} \left[F_{\nu, \sigma_0, \dots, \sigma_k}^{j, k, i_0, \dots, i_k} + (j \leftrightarrow k) \right] = 0, \quad k = 0, \dots, n. \quad (4.47)$$

It follows that the generic expression of the coefficients $E_{\sigma_0, \dots, \sigma_k}^{i_1, \dots, i_k}$ is:

$$E_{\sigma_0, \dots, \sigma_k}^{i_1, \dots, i_k} = G_{\sigma_0, \dots, \sigma_k}^{i_1, \dots, i_k} - y_{jk}^\nu F_{\nu, \sigma_0, \dots, \sigma_k}^{j, k, i_1, \dots, i_k} \quad (4.48)$$

where $G_{\sigma_0, \dots, \sigma_k}^{i_1, \dots, i_k}$ have the antisymmetry property (4.24) and verify

$$\partial_\nu^{jk} G_{i_0, \dots, i_k, \sigma_0, \dots, \sigma_k} = 0, \quad k = 0, \dots, n. \quad (4.49)$$

Substituting (4.48) into the expression (4.21) one obtains (4.45) with

$$\begin{aligned} \sigma = & \sum_{k=0}^n \frac{1}{k!} F_{\sigma_0, \dots, \sigma_k}^{i_0, \dots, i_k} dy_{i_0}^{\sigma_0} \wedge \omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_k} \wedge \theta_{i_1, \dots, i_k} + \\ & \sum_{k=0}^n \frac{1}{(k+1)!} G_{\sigma_0, \dots, \sigma_k}^{i_1, \dots, i_k} \omega^{\sigma_0} \wedge \omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_k} \wedge \theta_{i_1, \dots, i_k}. \end{aligned} \quad (4.50)$$

This finishes the proof. ■

It is plausible that the formalism presented in this section can be extended to Grassmann manifolds of arbitrary order $r > 2$. Some steps in this direction are contained in [22].

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